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SOME CONGRUENCES FOR PARTITION FUNCTIONS RELATED TO MOCK THETA FUNCTIONS $\omega(q)$ AND $\nu(q)$

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Abstract. Recently, Andrews, Dixit and Yee introduced partition functions $p_\omega(n)$, $p_\nu(n)$, smallest parts functions $\mathrm{spt}_\omega(n)$ and $\mathrm{spt}_\nu(n)$, which are closely related to the Ramanujan/Watson mock theta functions $\omega(q)$ and $\nu(q)$. In this paper we obtain new congruences modulo 20 for the partition function $p_\omega(n)$ which are extensions of the first explicit congruences for $\omega(q)$ given by Waldherr. We give a simple proof of Wang's conjecture for the smallest parts function $\mathrm{spt}_\omega(n)$ for $k=1$, which is a special case of Andrews' result $\mathrm{spt}_\omega(5n+3) \equiv 0 \pmod 5$.

1. Introduction

A partition of a positive integer n is a nonincreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_k$ such that $\lambda_1 + \lambda_2 \cdots + \lambda_k = n$. Let $p(n)$ be the number of partitions of n. Then, for example $p(5) = 7$. The seven partitions of 5 are 5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$, $1+1+1+1+1$. The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = \frac{1}{f_1},$$

where throughout this paper, we define $f_k := (q^k;q^k)_\infty = \prod_{m=1}^{\infty}(1-q^{mk})$.

The third order mock theta functions $\omega(q)$, $\nu(q)$ due to Ramanujan and Watson [**7, 10**], are defined by

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)^2_{n+1}},$$

$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}},$$

where, as customary, for any complex numbers a and $\mid q \mid < 1$,

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

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Andrews et al. [**1**], introduced some new partition functions which are closely related to the mock theta functions $\omega(q)$ and $\nu(q)$,

$$\sum_{n=1}^{\infty} p_\omega(n)q^n = q\,\omega(q), \tag{1.1}$$

where the function $p_\omega(n)$ counts the number of partitions of n in which all odd parts are less than twice the smallest part. For example, $p_\omega(5) = 6$. The six partitions of 5 it enumerates are 5, $4+1$, $3+2$, $2+2+1$, $2+1+1+1$, $1+1+1+1+1$.

$$\sum_{n=0}^{\infty} p_\nu(n)q^n = \nu(-q), \tag{1.2}$$

the function $p_\nu(n)$ counts the number of partitions of n in which the parts are distinct and all odd parts are less than twice the smallest part. For example, $p_\nu(5) = 3$. The three partitions of 5 it enumerates are 5, $4+1$, $3+2$. Andrews et al. [**1**], defined the associated smallest parts function $\mathrm{spt}_\omega(n)$, which counts the total number of smallest parts in the partitions enumerated by $p_\omega(n)$. For example, $\mathrm{spt}_\omega(5) = 12$. We have $\underline{5}$, $4+\underline{1}$, $3+\underline{2}$, $2+2+\underline{1}$, $2+\underline{1}+\underline{1}+\underline{1}$, $\underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1}$. They showed that the generating function of $\mathrm{spt}_\omega(n)$ is

$$\sum_{n=1}^{\infty} \mathrm{spt}_\omega(n)q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2(q^{n+1};q)_n(q^{2n+2};q^2)_\infty}. \tag{1.3}$$

Also in [**1**], $\mathrm{spt}_\nu(n)$ counts the number of smallest parts in the partitions enumerated by $p_\nu(n)$, but it is clear that $\mathrm{spt}_\nu(n) = p_\nu(n)$ since the parts are distinct, and each partition has a unique smallest part.
Using modular forms, Waldherr [**8**] obtained congruences equivalent to,

$$p_\omega(40n+28) \equiv 0 \pmod 5, \tag{1.4}$$

$$p_\omega(40n+36) \equiv 0 \pmod 5. \tag{1.5}$$

Recently, Wang [**9**] proved some new congruences

$$p_\omega(88n+r) \equiv 0 \pmod{11}, \quad \text{where } r \in \{20, 52, 60, 68, 84\}. \tag{1.6}$$

Also in his paper, he has made the following conjecture:
For any integers $k \geq 1$ and $n \geq 0$,

$$\mathrm{spt}_\omega\left(2\cdot 5^{2k-1}n + \frac{7\cdot 5^{2k-1}+1}{12}\right) \equiv 0 \pmod{5^{2k-1}}. \tag{1.7}$$

In Section 3 of this paper, we prove the following congruences:
For any integer $n \geq 0$,

$$p_\omega(40n+r) \equiv 0 \pmod{20}, \quad \text{where } r \in \{28, 36\},$$

which are extensions of (1.4) and (1.5). In Section 4, we give a simple proof of Wang's conjecture (1.7) for $k = 1$, which is also a special case of Theorem 6.2 in [**1**]. In the concluding section, we prove a number of arithmetic properties modulo 2 satisfied by $p_\omega(n)$ and $p_\nu(n)$. All of the proofs will follow from elementary generating function considerations and q–series manipulations.

2. Preliminaries

We require the following definitions and lemmas to prove the main results in the next three sections. For $|\, ab \,| < 1$, Ramanujan's general theta function $f(a,b)$ is defined as

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{2.1}$$

Using Jacobi's triple product identity [**4**, Entry 19, p. 35], (2.1) becomes

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.2}$$

The special cases of $f(a,b)$ are

$$\varphi(q) := f(q,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = (-q;q^2)^2_\infty(q^2;q^2)_\infty = \frac{f_2^5}{f_1^2 f_4^2}, \tag{2.3}$$

$$\psi(q) := f(q;q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1}. \tag{2.4}$$

Lemma 2.1. (*Andrews et al.* [**2**] *and Watson* [**10**, p. 63]) We have

$$\begin{aligned} &f(q^8) + 2q\omega(q) + 2q^3\omega(-q^4) \\ &= \frac{\phi(q)\phi^2(q^2)}{(q^4;q^4)^2_\infty} \\ &= F_0(q^4) + qF_1(q^4) + q^2F_2(q^4) + q^3F_3(q^4), \end{aligned} \tag{2.5}$$

where

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}$$

and

$$F_0(q) = \frac{\phi^3(q)}{(q;q)^2_\infty} = \frac{f_2^{15}}{f_1^8 f_4^6}, \qquad F_1(q) = 2\frac{\phi^2(q)\psi(q^2)}{(q;q)^2_\infty} = 2\frac{f_2^9}{f_1^6 f_4^2},$$

$$F_2(q) = 4\frac{\phi(q)\psi^2(q^2)}{(q;q)^2_\infty} = 4\frac{f_2^3 f_4^2}{f_1^4}, \qquad F_3(q) = 8\frac{\psi^3(q^2)}{(q;q)^2_\infty} = 8\frac{f_4^6}{f_1^2 f_2^3}.$$

Lemma 2.2. (*Wang* [**9**]) We have

$$\sum_{n=0}^{\infty} p_\omega(4n+1)\, q^n = \frac{f_2^9}{f_1^6 f_4^2}, \tag{2.6}$$

$$\sum_{n=0}^{\infty} p_\omega(8n+4)\, q^n = 4\frac{f_2^{10}}{f_1^9}, \tag{2.7}$$

$$\sum_{n=0}^{\infty} \mathrm{spt}_\omega(2n+1)\, q^n = \frac{f_2^8}{f_1^5}. \tag{2.8}$$

From the Binomial Theorem, for any positive integer, k,

$$f_k^5 \equiv f_{5k} \pmod 5, \tag{2.9}$$

$$f_1^{2^k} \equiv f_2^{2^{k-1}} \pmod{2^k}. \tag{2.10}$$

3. Congruences Modulo 20 for $p_\omega(n)$

In this section we prove two congruences modulo 20 for $p_\omega(n)$.

Theorem 3.1.

$$\sum_{n=0}^{\infty} p_\omega(40n+12)q^n \equiv 16\frac{f_5 f_2^2}{f_1^2} \pmod{20}. \tag{3.1}$$

Proof. From (2.7), we have

$$\sum_{n=0}^{\infty} p_\omega\,(8n+4)\,q^n = 4\frac{f_2^{10} f_1}{f_1^{10}}. \tag{3.2}$$

Using (2.9), we obtain

$$\sum_{n=0}^{\infty} p_\omega\,(8n+4)\,q^n \equiv 4\frac{f_{10}^2 f_1}{f_5^2} \pmod{20}. \tag{3.3}$$

Recall Ramanujan's beautiful identity [**3**, Eqn. (7.4.1), p. 161]:

$$f_1 = f_{25}\left(R^{-1}(q^5) - q - q^2 R(q^5)\right), \tag{3.4}$$

where

$$R(q) = \frac{f(-q,-q^4)}{f(-q^2,-q^3)}.$$

Substituting (3.4) in (3.3), we obtain

$$\sum_{n=0}^{\infty} p_\omega\,(8n+4)\,q^n \equiv 4\frac{f_{10}^2 f_{25}}{f_5^2}\left(R^{-1}(q^5) - q - q^2 R(q^5)\right) \pmod{20}. \tag{3.5}$$

Equating the coefficients of q^{5n+1} from both sides of (3.5), dividing both sides by q and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} p_\omega\,(40n+12)\,q^n \equiv -4\frac{f_2^2 f_5}{f_1^2} \pmod{20}. \tag{3.6}$$

□

Theorem 3.2. *For all $n \geq 0$,*

$$p_\omega(40n+28) \equiv 0 \pmod{20}, \tag{3.7}$$

$$p_\omega(40n+36) \equiv 0 \pmod{20}. \tag{3.8}$$

Proof. Comparing the coefficients of q^{5n+3} from both sides of (3.5), we obtain (3.7). Similarly comparing the coefficients of q^{5n+4} from both sides of (3.5), we have (3.8). □

Waldherr's results, given earlier, follow from (3.7) and (3.8).

4. Proof of Wang's Conjecture

In the next theorem, we prove Wang's conjecture (1.7) for $k = 1$.

Theorem 4.1. For all $n \geq 0$,

$$\mathrm{spt}_\omega\,(10n+3) \equiv 0 \pmod{5}. \tag{4.1}$$

Proof. From (2.8), we have

$$\sum_{n=0}^{\infty} \mathrm{spt}_\omega\,(2n+1)\,q^n = \frac{f_2^5 f_2^3}{f_1^5}. \tag{4.2}$$

Using (2.9), we obtain

$$\sum_{n=0}^{\infty} \mathrm{spt}_\omega\,(2n+1)\,q^n \equiv \frac{f_{10} f_2^3}{f_5} \pmod{5}. \tag{4.3}$$

From Jacoi's Identity [**6**], we have

$$f_1^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{(n^2+n)/2} \equiv f(-q^{10}, -q^{15}) - 3q f(-q^5, -q^{20}) \pmod{5}. \tag{4.4}$$

Replacing q by q^2 in (4.4) and then substituting the resultant identity in (4.3), we obtain

$$\sum_{n=0}^{\infty} \mathrm{spt}_\omega\,(2n+1)\,q^n$$
$$\equiv \frac{f_{10}}{f_5} \left(f(-q^{20}, -q^{30}) - 3q^2 f(-q^{10}, -q^{40}) \right) \pmod{5}. \tag{4.5}$$

Comparing the coefficients of q^{5n+1} from both sides of (4.5), we arrive at (4.1). □

Again, on comparing the coefficients of q^{5n+3} and q^{5n+4} from both sides of (4.5), we obtain

Theorem 4.2. *(*Andrews et al.[**1**] *)* For all $n \geq 0$,

$$\mathrm{spt}_\omega\,(10n+7) \equiv 0 \pmod{5}, \tag{4.6}$$

$$\mathrm{spt}_\omega\,(10n+9) \equiv 0 \pmod{5}. \tag{4.7}$$

5. Congruences for $p_\omega(n)$ and $p_\nu(n)$

In this section, we prove a number of arithmetic properties modulo 2 satisfied by $p_\omega(n)$ and $p_\nu(n)$.

Theorem 5.1. *For all $n \geq 0$,*

$$p_\omega(16n+5) \equiv 0 \pmod 2, \tag{5.1}$$

$$p_\omega(16n+9) \equiv 0 \pmod 2. \tag{5.2}$$

Proof. From (2.6), we have

$$\sum_{n=0}^{\infty} p_\omega(4n+1)\, q^n = \frac{f_2^9}{f_1^6 f_4^2}. \tag{5.3}$$

Using (2.10), we obtain

$$\sum_{n=0}^{\infty} p_\omega(4n+1)\, q^n \equiv \frac{f_2^9}{f_2^3 f_4^2} = \frac{f_2^6}{f_4^2} \equiv \frac{f_4^3}{f_4^2} = f_4 \pmod 2. \tag{5.4}$$

Comparing the coefficients of q^{4n+1}, from both sides of (5.4), we obtain (5.1). Similarly comparing the coefficients of q^{4n+2}, from both sides of (5.4), we arrive at (5.2). □

Again, on comparing the coefficients of q^{4n+3} from both sides of (5.4), we have

$$p_\omega(16n+13) \equiv 0 \pmod 2, \tag{5.5}$$

congruences (5.5) is also true for moduli 4, confirmed by Andrews et al. [**2**, Th. 3.4].

Theorem 5.2. *For all $n \geq 0$,*

$$p_\omega(16n+1) \equiv \begin{cases} 1 \pmod 2 & \text{if } n \text{ is a pentagonal number}, \\ 0 \pmod 2 & \text{otherwise}. \end{cases} \tag{5.6}$$

Proof. Comparing the coefficients of q^{4n} from both sides of (5.4) and then replacing q^4 by q, we obtain

$$\sum_{n=0}^{\infty} p_\omega(16n+1)\, q^n \equiv f_1 \pmod 2. \tag{5.7}$$

Using Euler's Pentagonal Number Theorem [**3**, p. 12], we have

$$\sum_{n=0}^{\infty} p_\omega(16n+1)\, q^n \equiv \sum_{k=-\infty}^{\infty} q^{\frac{k(3k+1)}{2}} \pmod 2. \tag{5.8}$$

□

Theorem 5.3. For all $\alpha, n \geq 0$,

$$p_\omega\left(2^{2\alpha+4}n + \frac{14 \cdot 2^{2\alpha}+1}{3}\right) \equiv 0 \pmod 2, \tag{5.9}$$

$$p_\omega\left(2^{2\alpha+4}n + \frac{26 \cdot 2^{2\alpha}+1}{3}\right) \equiv 0 \pmod 2. \tag{5.10}$$

Proof. Equating the coefficients of q^{4n+3} from both sides of (2.5), we have

$$\sum_{n=0}^{\infty} p_\omega(4n+3)q^{4n+3} + q^3\omega(-q^4) = \frac{1}{2}q^3 F_3(q^4), \tag{5.11}$$

which yields

$$\sum_{n=1}^{\infty} (p_\omega(4n-1) - (-1)^n p_\omega(n))q^n = \frac{1}{2}qF_3(q) = 4q\frac{f_4^6}{f_1^2 f_2^3}. \tag{5.12}$$

Thus, for all $n \geq 1$,

$$p_\omega(4n-1) \equiv p_\omega(n) \pmod 2. \tag{5.13}$$

From (5.1), we have

$$p_\omega(16n+5) \equiv 0 \pmod 2, \tag{5.14}$$

which is the $\alpha = 0$ case of (5.9). Suppose (5.9) is true for some $\alpha \geq 0$. Substituting (5.9) in (5.13), we obtain

$$p_\omega\left(2^{2(\alpha+1)+4}n + \frac{14 \cdot 2^{2(\alpha+1)} + 1}{3}\right) \equiv 0 \pmod 2, \tag{5.15}$$

which is (5.9) with $\alpha + 1$ for α. This completes the proof of (5.9) by mathematical induction. Note that (5.2) is the $\alpha = 0$ case of (5.10). Using (5.13) and mathematical induction, we obtain (5.10). □

Theorem 5.4. For all $\alpha, n \geq 0$,

$$p_\nu\left(2^{2\alpha+5}n + \frac{14 \cdot 2^{2\alpha+1} - 1}{3}\right) \equiv 0 \pmod 2, \tag{5.16}$$

$$p_\nu\left(2^{2\alpha+5}n + \frac{26 \cdot 2^{2\alpha+1} - 1}{3}\right) \equiv 0 \pmod 2. \tag{5.17}$$

Proof. From [**5**, p.62, Eqn. (26.88)], we have

$$\nu(-q) = q\omega(q^2) + (-q^2;q^2)_\infty^3 (q^2;q^2)_\infty. \tag{5.18}$$

Using (1.2) in (5.18), we obtain

$$\sum_{n=0}^{\infty} p_\nu(n)q^n = q\omega(q^2) + (-q^2;q^2)_\infty^3 (q^2;q^2)_\infty. \tag{5.19}$$

Extracting the coefficients of q^{2n+1} from both sides of (5.19), we obtain

$$\sum_{n=1}^{\infty} p_\nu(2n-1)q^{2n} = q^2\omega(q^2) \tag{5.20}$$

Thus, we have

$$p_\nu(2n-1) = p_\omega(n). \tag{5.21}$$

Using (5.21) in (5.9) and (5.10), we obtain (5.16) and (5.17).

□

We close this section by briefly noting the following corollaries of Theorem 5.2 and Theorem 3.2.

Corollary 5.5. *For all $n \geq 0$,*

$$p_\nu(32n+1) \equiv \begin{cases} 1 \pmod 2 & \text{if } n \text{ is a pentagonal number,} \\ 0 \pmod 2 & \text{otherwise.} \end{cases} \tag{5.22}$$

Proof. This follows from (5.6) and (5.21). □

Corollary 5.6. *For all $n \geq 0$,*

$$p_\nu(80n+55) \equiv 0 \pmod{20}, \tag{5.23}$$

$$p_\nu(80n+71) \equiv 0 \pmod{20}. \tag{5.24}$$

Proof. The above congruences follow from (3.7), (3.8) and (5.21). □

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