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AN ABSTRACT ALGEBRAIC-TOPOLOGICAL APPROACH TO THE NOTIONS OF A FIRST AND A SECOND DUAL SPACE III

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Abstract. Here we continue to develop a concept, that generalizes the idea of the second dual space of a normed vector space in a fairly general way. As in the part before, the main tool is to recognize the "first dual" as a means to the end of the second dual. Especially, we will easily prove here some essential statements on embeddings of noncommutative C^*-algebras in their second dual, as whose analogues are known in the commutative setting.

Contents

1. Introduction

Duality principles occur in many forms in topology and functional analysis. One of them is the concept of the dual space X^d for a normed vector space X. From a more abstract point of view, this is just the idea to attend to each of a class of some spaces X a space of morphisms into a fixed range space - in fact, a very special function space. Really fascinating is the power that this concept developes if it is applied twice: suddenly there are very friendly embeddings of the original space X into its second dual X^{dd} !

In order to apply the same process of dualization twice, it seems to be necessary, to transfer the entire kind of structure of the base space X to its dual X^d . Unfortunately, this is impossible in much other cases than normed vector spaces - because the algebraic structure hardly struggles.

However, the most interesting results in our perception here, are concerned with the *second* dual X^{dd} . So, in this context we propose to relinquish the idea of *one concept* of duality that has to be applied *twice* to bring out a powerful tool. Instead, we described in the former two parts [**1**], [**2**] of this little series a fairly general procedure to get very appropriate second duals X^{dd} using just such first duals, whose construction is no longer solicitous to reproduce the structure of the original space X in the function space X^d but to enable such a structure in the function space X^{dd} with the same range, but with domain X^d .

Here in part III we want to find out, how this concept works for (even noncommutative) C^*-algebras, and we generalize the solution of the representation problem (as

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given for the special noncommutative matrix-algebras $M_n(\mathbb{C})$ in [**2**]) to arbitrary C^*-algebras.

2. The Hierarchy of First Dual Spaces for Banach-Algebras and C^*-Algebras

Let X be a C^*-algebra; $X' := \{h : X \to \mathbb{C} \mid h$ linear and continuous$\}$ the (first) dual space of the Banach space $(X, ||.||)$. By definition 3.2 of [**1**] (see also definition 2.1 of [**4**]) we get:

$$X^d = \{h : X \to \mathbb{C} \mid h \text{ linear, continuous, multiplicative}\}$$

as dual space of the Banach algebra and

$$X^{d_*} = \{h : X \to \mathbb{C} \mid h \text{ linear, continuous, multiplicative, involutary}\}$$

as dual space of the C^*-algebra X. Now we find at once: $X^{d_*} \subseteq X^d \subseteq X'$.

The C^*-algebra $\mathbb{C}$ has a zero element 0, hence there exists a zero-functional $\underline{0} \in X^{d_*}$, and the question arises: has X^{d_*} non-trivial elements, too, meaning $X^{d_*} \setminus \{\underline{0}\} \neq \emptyset$?

We know that X' has many functionals and if X is a commutative Banach algebra with unit e then X^d has many elements too. But otherwise for the matrix algebras $M_n(\mathbb{C})$, $n > 1$, non-commutative C^*-algebras with unit, hold $(M_n(\mathbb{C}))^d = \{\underline{0}\}$ (see [**2**], 3.5, for instance).

Now let

$$X^d_{alg} = \{h : X \to \mathbb{C} \mid h \text{ linear and multiplicative }\}$$
$$X^{d_*}_{alg} = \{h : X \to \mathbb{C} \mid h \text{ linear, multiplicative and involutory }\},$$

where we only incorporate the algebraic operations of X and $\mathbb{C}$ and not the topologies on these spaces. We find

Proposition 2.1.

$$X^d \subseteq X^d_{alg},\ X^{d_*} \subseteq X^{d_*}_{alg}$$

and hence

$$X^d = X^d_{alg},\ X^{d_*} = X^{d_*}_{alg}.$$

Proof. Linear and multiplicative functionals $h : X \to \mathbb{C}$ are bounded and hence continuous. □

For our dual spaces we clearly define algebraic operations pointwise, for instance, $h_1, h_2 \in X^d, \forall x \in X : (h_1h_2)(x) := (h_1(x))(h_2(x))$ or $h \in X^{d_*}, \forall x \in X : h(x^*) := \overline{h(x)}$.

Concerning the algebraic structure of our spaces we get:

(1) For the Banach space X, X is a $\mathbb{K}$-vector space and X' is a vector space again.

(2) For the dual spaces X^d, X^{d_*}, X is a $\mathbb{K}$-algebra, especially a $\mathbb{C}$-algebra, but in general X^d, X^{d_*} are not algebras, for example, if $h_1, h_2 \in X^d$ then h_1h_2 mostly will fail to belong to X^d, since it may not be additive.

As mentioned already above, also in this situation we have in [**1**] defined a convenient second dual space X^{dd} (definitions 4.1, 4.2).

By the canonical map J (see [**1**]) we can embed X into X^{dd}. This was for commutative algebras done by Gelfand and Naimark long times ago, we only give here an interpretation via the notions of a first and a second dual space as defined in [**1**].

For C^*-algebras we still mention two (known) special facts. If X, Y are C^*-algebras, then according to definition [**1**], 3.2.,

$$\begin{aligned} X^{d_*} &= \{h \in Y^X |\ h \text{ linear, multiplicative, involutary, continuous } \} \\ &= \{h \in Y^X |\ h \text{ is }^*\text{-homomorphism and continuous}\} \end{aligned}$$

is the dual space of X *w.r.t.* Y.

Let again be $X^{d_*}_{alg} = \{h \in Y^X |\ h \text{ is }^*\text{-homomorphism}\}$.

Proposition 2.2. *Let X, Y be unital C^*-algebras. Then hold*
(a) $X^{d_*} = X^{d_*}_{alg}$
(b) *If $Y = \mathbb{C}$ and X is commutative, then $X^{d_*} = X^d$.*

Proof. (a) By our assumptions, a *-homomorphism is continuous (see [**2**], prop. 3.1; [**5**], prop. 5.2).

(b) We must show: $X^d \subseteq X^{d_*}$, i.e. if $h \in X^d$ is linear and multiplicative, then h is involutary, too. If $h = 0$ is the zero functional, then it is involutary; so, let $h \in X^d, h \neq 0$. We have $\forall x \in X : x = x_1 + ix_2$, where $x_1, x_2 \in X$ are self-adjoint; now, if $y \in X$ is self-adjoint and $\sigma(y)$ is the spectrum then $\sigma(y) \subseteq \mathbb{R}$ holds and for an arbitrary element $x \in X : \sigma(x) = \{h(x)|\ h \in X^d, h \neq 0\}$. Hence we get $x = x_1 + ix_2 \implies x^* = x_1^* - ix_2^* = x_1 - ix_2 \implies h(x^*) = h(x_1) - ih(x_2) \in \mathbb{C}$ and thus $h(x^*) = \overline{h(x_1) + ih(x_2)} = \overline{h(x)}$.

□

Corollary 2.3. *By the assumtions of proposition 2.2 (b) we get: Since X^d has enough elements, X^{d_*} has also enough elements.*

Now we present some useful results.

Proposition 2.4. *Let X, Y be Banach algebras or C^*-algebras, respectively, and X^d or X^{d_*} the dual spaces of X with respect to Y; for Y^X we consider the pointwise topology τ_p, where τ_p coincides with the Tychonoff product topology, if we identify*

$$Y^X \equiv \prod_{x \in X} Y_x ,$$

where $Y_x := Y$ for all $x \in X$. Then hold
(a) *X^d_{alg} and $X^{d_*}_{alg}$ are closed subsets of (Y^X, τ_p).*
(b) *If X, Y are unital, then X^{d_*} is a closed subset of (Y^X, τ_p).*
(c) *If $Y = \mathbb{C}$ and X is commutative and unital, then X^d is a closed subset of (Y^X, τ_p).*
(d) *If $Y = \mathbb{C}$ and X is a commutative and unital C^*-algebra, then X^{d_*} is a closed subset of $(Y^X, \tau_p) = (\mathbb{C}^X, \tau_p)$.*
Note, that all duals X^d , X^{d_*} contain 0.

Proof. (a) Y is a metric space and hence it is Hausdorff, moreover, all algebraic operations of Y, (including the involution in case, that Y is a C^*-algebra) are continuous. Thus by [**4**], proposition 3.1 (see also [**1**], 4.9) X^d_{alg} and $X^{d_*}_{alg}$ are closed in (Y^X, τ_p).
Assertions (b), (c) and (d) now follow from proposition 2.2. □

Concluding remark: our dual spaces are subsets of the Banach space dual X'. In X' we can define the operator norm: $\forall h \in X' : ||h|| := \sup\{|h(x)| \mid ||x|| \leq 1\}$, and hence for X^d , X^{d_*} we can use this norm, too.

3. Simultaneous Proof of the Theorems of Gelfand and Gelfand-Naimark for Commutative Algebras

Let X be a commutative $\mathbb{C}$-Banach algebra with unit e and X^d the dual space of X. From section 2 we know that in general X^d (with pointwise defined algebraic operations) is not an algebra again. Hence according to definition 4.1. of [**1**], X^d has the defect D. Thus by definition 4.2. of [**1**] we get the second dual space of X as

$$X^{dd} = \left(C\left((X^d, \tau_p), (\mathbb{C}, \tau_{|.|})\right), \mu \right) ,$$

where μ is a topology for $C(X^d, \mathbb{C})$ with $\tau_p \subseteq \mu$ and $\tau_{|.|}$ denotes the Euclidian topology on $\mathbb{C}$.

We will show now, that (X^d, τ_p) is compact Hausdorff, but we don't use here the Alaoglu theorem as is usually done.

Proposition 3.1. *Let X be an unital commutative $\mathbb{C}$-Banach algebra. Then (X^d, τ_p) is a compact and Hausdorff topological space.*

Proof. $(\mathbb{C}^X, \tau_p)$ is Hausdorff, because $\mathbb{C}$ is. For $x \in X$ let $\sigma(x)$ be the spectrum and let $\tilde{\sigma}(x) := \sigma(x) \cup \{0\}$; then $\tilde{\sigma}(x)$ is compact, too, and of course Hausdorff. By the Tychonoff theorem $\prod_{x \in X} \tilde{\sigma}(x)$ with product topology is a compact Hausdorff subspace of $(\mathbb{C}^X, \tau_p)$. Using $\forall x \in X : \sigma(x) \supseteq \{h(x) |\ 0 \neq h \in X^d\}$ and identifying $\forall h \in X^d$ just $h \equiv (h(x))_{x \in X}$, we recognize X^d as a subspace of $\prod_{x \in X} \tilde{\sigma}(x)$, which is closed in $(\mathbb{C}^X, \tau_p)$ by Proposition 2.4 (c), and so is closed in $\prod_{x \in X} \tilde{\sigma}(x)$, consequently it is compact, and of course Hausdorff. □

Corollary 3.2. *Let X be a commutative unital C^*-algebra. Then (X^{d_*}, τ_p) is a compact Hausdorff topological space.*

Proof. Follows from proposition 3.1 since by 2.2 (b) $X^{d_*} = X^d$ holds. □

Corollary 3.3. *Let X be a commutative $\mathbb{C}$-Banach algebra or a C^*-algebra with unit. Then hold*

(a) $X^{dd} = C\left((X^d, \tau_p), \mathbb{C}\right) = C_b\left((X^{d_*}, \tau_p), \mathbb{C}\right)$ *- the bounded continuous functions. Hence we chose for μ the uniform topology of $C(X^d, \mathbb{C})$, which is generated by the supremum norm $||.||_{sup}$.*

(b) $\left((C_b(X^d, \tau_p), \mathbb{C}), ||.||_{sup}\right)$ *is a commutative Banach algebra or C^*-algebra with unit.*

(c) *The zero functional 0 is an isolated point of (X^d, τ_p) and of (X^{d_*}, τ_p), respectively and $(X^d \setminus \{0\}, \tau_p), (X^{d_*} \setminus \{0\}, \tau_p)$ are compact Hausdorff, too.*

Proof. (a) is evident, (b) follows from proposition 4.2 of [**1**], and (c) follows from lemmas 4.3, 4.4 of [**1**]. □

Remark 3.4 It is appropriate to exclude the zero functional from X^d, since for instance $ker(0) = X$ and X is not a proper ideal. Compare also section 4.7 of [**1**], lemma 4.2.: furtheron we will use the set $X^d \setminus \{0\}$ and denote it by the symbol X^d again. "$0 \notin X^d$" emphasizes, that X^d is not an algebra.

As already mentioned, according to [**1**] we denote the canonical map from X to X^{dd} (Gelfand map w.r.t. Banach algebras) by $J : X \to X^{dd}$, $Jx = \omega(x, \cdot)$, $\omega(x, \cdot) : X^d \to \mathbb{C} : \forall h \in X^d : \omega(x, \cdot)(h) = \omega(x, h) = h(x)$, with the *evaluation map* ω.

J means $Jx = \omega(x, \cdot) : X^d \to Y$, if X, Y are Banach algebras and X^d, X^{dd} mean the first an the second dual space with respect to Y, as was developed in [**1**].

We still come back to some known facts and we will prove a lemma.

Proposition 3.5. 1. *Let X be a commutative Banach algebra with unit. Then holds:*

$$\forall x \in X : ||Jx||_{sup} = r(x) ,$$

where $r(x)$ denotes the spectral radius.

2. *Now let X be a C^*-algebra with unit. We have:*
   (a) $\forall x \in X$, *x normal:* $||x^2|| = ||x||^2$.
   (b) *If x is normal and $n \in \mathbb{N}, n \geq 1$, then x^n is normal.*
   (c) *If x is normal and $k \in \mathbb{N}, k \geq 1$, then* $||x^{2^k}|| = ||x||^{2^k}$.
   (d) *Using (c) and the spectral radius formula, one gets:* $r(x) = ||x||$.

Lemma 3.6. *Let X, Y be sets and s, t unitary algebraic operations in X and Y, respectively:*

$$\forall x \in X : s : x \mapsto x^s \in X,$$
$$\forall y \in Y : t : y \mapsto y^t \in Y \ .$$

Now let $\emptyset \neq A \subseteq X$ be s-closed, i.e. $\forall x \in A : x^s \in A$; and let $h : X \to Y$ be a s-t-homomorphism, i.e. $\forall x \in X : h(x^s) = (h(x))^t$.
Then $h(A)$ is t-closed.

Proof. $\forall y \in h(A) : \exists x \in A : h(x) = y$; $x \in A \implies x^s \in A \implies h(x^s) \in h(A)$. $h(x^s) = (h(x))^t = y^t \implies y^t \in h(A)$. □

Corollary 3.7. *Let X, Y be algebras, each with an involution *, and let $h : X \to Y$ be a *-preserving map, i.e. $\forall x \in X : h(x^*) = (h(x))^*$. Then $h(X) \subseteq Y$ is self-adjoint, i.e. $\forall y \in h(X) : y^* \in h(X)$.*

Theorem 3.8 (Gelfand; Gelfand-Naimark)**.** *Let X be a commutative Banach algebra or a C^*-algebra, respectively, with unit e.*

1. *If X is just a Banach algebra, then (X^d, τ_p) is a compact and Hausdorff topological space.*
   If X is a C^*-algebra, then $(X^{d_*}, \tau_p) = (X^d, \tau_p)$ is a compact and Hausdorff topological space.
2. $X^{dd} = \left(C_b\left((X^d, \tau_p),\ \mathbb{C}\ \right), ||.||_{sup}\right) = \left(C_b\left((X^{d_*}, \tau_p),\ \mathbb{C}\ \right), ||.||_{sup}\right)$ *is a commutative Banach algebra, if X is. It is a commutative C^*-algebra, if X is. X^{dd} has the constant function $\mathbb{1}$ as unit.*
3. *For the canonical map $J : X \to \mathbb{C}^{\,X^d}$ holds $J(X) \subseteq X^{dd}$ and $J : X \to X^{dd}$ is an algebra homomorphism, if X is a Banach algebra. It is also a $*$-homomorphism, if X is a C^*-algebra.*
4. $\forall x \in X : ||Jx||_{sup} \leq ||x||$ *holds, if X is a Banach algebra.*
   $\forall x \in X : ||Jx||_{sup} = ||x||$ *holds, if X is a C^*-algebra.*
5. *J is uniformly continuous and hence continuous.*
6. *$J(X)$ is a subalgebra of X^{dd} , if X is a Banach algebra. $J(X)$ is a $\tau_{||.||_{sup}}$-closed and self-adjoint subalgebra of X^{dd} , if X is a C^*-algebra.*
7. $J(e) = \mathbb{1} \in J(X) \subseteq C_b\left((X^d, \tau_p),\ \mathbb{C}\ \right)$.
8. *$J(X)$ separates the points of $X^d = X^{d_*}$.*
9. *If X is a commutative C^*-algebra with unit e, then hold:*
   (a) $J(X) = X^{dd} = C_b\left((X^d, \tau_p),\ \mathbb{C}\ \right)$,
   (b) *J yields an isometry and an isomorphy between X and $(X^{dd}, ||.||_{sup})$.*

Proof. 1. comes from proposition 3.1 and corollary 3.3

For 2. compare the proof of corollary 3.3.

3. Corollary 4.1. of [**1**] shows that $J(X) \subseteq X^{dd}$ holds and by the general homomorphy theorem for the canonical map J, theorem 4.1. of [**1**], J has the asserted properties[1].

[1]Don't worry about the number 4.1.: in [**1**] occur (in chronological order) a *definition* 4.1., a *subsection* 4.1., a *lemma* 4.1., a *corollary* 4.1., a *theorem* 4.1. and a *proposition* 4.1. - probably caused by a forgotten optional parameter in the definition of the TEX environments for such statements. We prefer to think of it as a record attempt.

4. It is well-known that $r(x) \leq ||x||$ holds, and by proposition 3.5 1. holds $||Jx||_{sup} = r(x)$; in case of a C^*-algebra by 2.(d) of proposition 3.5 we have $r(x) = ||x||$ and hence $||Jx||_{sup} = ||x||$.

5. Since especially J is linear:
$\forall x, y \in X : ||Jx - Jy||_{sup} = ||J(x-y)||_{sup} \leq ||x - y||$.

6. Since J is a homomorphism, it is evident, that $J(X)$ is a subalgebra of X^{dd} ; if X is a C^*-algebra, then by 4. J is an isometric map and thus J is injective (see for instance corollary 3.2. of [**2**]). Being an isometry, J is also an uniform isomorphism. Completeness is an invariant of uniform isomorphy, yielding that $J(X)$ is a complete subspace of $(X^{dd}, ||.||_{sup})$ and hence a closed subspace, too, because $(X^{dd}, ||.||_{sup})$ is Hausdorff. Self-adjointness follows from corollary 3.7.

7. All functionals $h \in X^d = X^{d_*}$ are multiplicative: $\forall h \in X^d : h(e) = 1 \in \mathbb{C}$. Now, $J(e) = \omega(e, \cdot) : \forall h \in X^d : \omega(e, \cdot)(h) = h(e) = 1 \implies J(e) = \mathbb{1} \in J(X)$.

8. We know that $X^d = X^{d_*}$ has enough elements: $\forall h_1, h_2 \in X^d : h_1 \neq h_2 \implies \exists x \in X : h_1(x) \neq h_2(x)$. $Jx \in J(X)$ and $Jx(h_1) = h_1(x) \neq h_2(x) = Jx(h_2)$. Thus $J(X)$ separates the points of X^d .

9. (a) Since $J(X)$ has the properties mentioned in 6., 7. and 8. the Stone-Weierstrass-theorem yields $J(X) = C_b\left((X^{d_*}, \tau_p), \mathbb{C}\right) = X^{dd}$. (Remark: Since we know that (X^{d_*}, τ_p) is compact Hausdorff, we also can write $C_b\left((X^d, \tau_p), \mathbb{C}\right) = C\left((X^d, \tau_p), \mathbb{C}\right)$.)

(b) We know that J is injective and a *-homomorphism. Then $J^{-1} : X^{dd} \to X$, $J^{-1}(X^{dd}) = X$, is a *-homomorphism, too. □

4. Representation of Unital Noncommutative C^*-Algebras by Closed Subsets of Spaces of Bounded Continuous Functions

Let X, Y be unital C^*-algebras, let X be noncommutative. If Y is commutative, then it is possible that the dual space X^d of X w.r.t. Y contains only the zero function. For example, at the beginning of section 2 we mentioned that for $X = M_n(\mathbb{C})$ and $Y = \mathbb{C}$ follows $(M_n(\mathbb{C}))^d = \{0\}$. To avoid such a situation we set $Y = X$. In [**2**] we considered a concrete example: we defined $X = Y = M_n(\mathbb{C})$. Here we study the general case.

According to definition 3.2 of [**1**] the dual space of X (w.r.t. X) is given by:

Definition 4.1. *Let X be a noncommutative C^*-algebra with unit e.*

$$\begin{aligned} X^{d_*} &:= \{h : X \to X |\ h \textit{ is a } {}^*\textit{-homomorphism and is continuous}\} \\ &= \{h : X \to X |\ h \textit{ is a } {}^*\textit{-homomorphism}\}. \end{aligned}$$

In X^{d_*} the addition of two operators and multiplication of an operator from X^{d_*} with a scalar $\lambda \in \mathbb{C}$ we must define pointwise. But for the multiplication of operators we have two possibilities:
(a) the composition of two operators $h_1, h_2 \in X^{d_*}$: $h_1 h_2 := h_1 \circ h_2$, or
(b) we define the product pointwise.

What can we say about the space X^{d_*} ?

At first we need two well-known results. By proposition 3.1. of [**2**], proposition 9.1., theorem 11.1 of [**6**] we get:

Proposition 4.2. *Let X, Y be unital C^*-algebras, $h : X \to Y$ a *-homomorphism. Then hold*
(1) $\forall x \in X : ||h(x)|| \leq ||x||$.
(2) *$h(X)$ is a C^*-subspace of Y.*

Corollary 4.3. *Let X, Y be unital C^*-algebras, $h : X \to Y$ a *-homomorphism. Then the operator norm $||h||$ of h exists and $||h|| \leq 1$ holds.*

Proof. $h \in L(X,Y) \implies ||h||$ exists and $||h|| = \sup\{||h(x)||| \; x \in X, ||x|| \leq 1\}$ $\leq \sup\{||x||| \; ||x|| \leq 1\} \leq 1$. □

Now we consider X^{d_*}.

(a) $X^{d_*} \neq \emptyset$: the identity operator $\mathbb{1}_X : X \to X$ belongs to X^{d_*}, and also each further *-automorphism: $\forall u \in X, u$ unitary : $h_u : \forall x \in X : h_u(x) := uxu^*$ is an element of X^{d_*}. That means, if X has enough unitary elements, then X^{d_*} has enough elements. Here $\mathbb{1}_X = h_e$.

(b) X^{d_*} is not a vector space: let $h \in X^{d_*}, h \neq 0$, then by corollary 4.3 we find $0 < ||h|| \leq 1$. If X^{d_*} would be a vector space, then $\frac{2}{||h||}h \in X^{d_*}$, but $||\frac{2}{||h||}h|| = 2 > 1$, a contradiction to corollary 4.3.

(c) Moreover, as we know (see again Lemma 4.2. of [**1**]), we must use $X^{d_*} \setminus \{0\}$ instead of X^{d_*}, and concerning notations we again identify $X^{d_*} \equiv X^{d_*} \setminus \{0\}$.

(d) Following our general duality approach, we choose the pointwise defined multiplication in X^{d_*}. It turns out by (b) that X^{d_*} has the defect D according to definition 4.1. of [**1**]. Hence by definition 4.2. of [**1**] we get the second dual space as $X^{dd} := \left(C\left((X^{d_*}, \tau_p), X\right), \; \mu\right)$. Which topology μ should we use in X^{dd}? The answer is given by the following lemma.

By corollary 4.1. of [**1**] we have $J(X) \subseteq X^{dd}$, but we get more:

Lemma 4.4. *Let X be a unital (noncommutative) C^*-algebra. Then $J(X) \subseteq C_b\left((X^{d_*}, \tau_p), X\right)$ holds. Hence for μ can take the uniform topology, which is generated by the* sup*-norm.*

Proof. $J(X) = \{\omega(x, \cdot)| \; x \in X\} \subseteq X^{X^{d_*}}$. We will show that $\omega(x, \cdot)(X^{d_*})$ is bounded in X for every $x \in X$: we have $\forall h \in X^{d_*} : \omega(x, \cdot)(h) = \omega(x, h) = h(x)$, implying $||\omega(x, \cdot)(h)|| = ||h(x)|| \leq ||x||$ by proposition 4.2, since each h is a *-homomorphism. So, $\omega(x, \cdot)(X^{d_*})$ is a subset of the ball with radius $||x||$ in X, meaning that $\omega(x, \cdot)$ is a (continuous) bounded map. □

Theorem 4.5. *Let X be a noncommutative C^*-algebra with unit e. Then hold*

1. $X^{dd} := \left(C\left((X^{d_*}, \tau_p), X\right), \; ||.||_{sup}\right)$ *is a (noncommutative) C^*-algebra with unit $\mathbb{1}$.*
2. $J : X \to X^{dd}$ *is a *-homomorphism.*
3. $J(X)$ *is a C^*- subspace of $(X^{dd}, ||.||_{sup})$.*
4. J *is an isometric map from X onto $J(X)$.*
5. J *is uniformly continuous and hence continuous.*
6. $J(e) = \mathbb{1}$.
7. $J(X)$ *separates the points of X^{d_*}.*
8. J *is an isometric and isomorphic map from X onto $J(X)$.*

Proof. 1. Follows from proposition 4.2. of [**1**].

2. The general homomorphy theorem 4.1 of [**1**] ensures that J is a *-homomorphism.

3. Follows from proposition 4.2(2).

4. By proposition 4.5. of [**1**], J is injective if and only if X^{d_*} separates the points of X. But we have $\mathbb{1}_X \in X^{d_*}$. Now, an injective *-homomorphism between two C^*-algebras is isometric (see, for instance, corollary 3.2. of [**2**]).

5. An isometric map is automatically uniformly continuous.

For 6., 7. compare the proof of theorem 3.8

The proof of 8. we obtain from 1., 2., 4., 3. and 6.. □

Question 4.6. *Which topological properties has (X^d, τ_p)?*

Question 4.7. *$J(X)$ is uniformly closed in the second dual. Is the pointwise closedness of $J(X)$ equivalent to X being a v.-Neumann-algebra?*

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