A Comparison of Methods for Portfolio Optimization

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Abstract

Fund managers face the problem of allocating investor's funds to different securities such that the return from the investment is maximized while the risk of the investment is kept to a minimum. This is known as the portfolio optimization problem. The traditional Markowitz model computes an investment portfolio that maximizes the expected return for a given level of risk. The model assumes that the returns from individual securities are independent over time. However, this assumption is not valid for all securities. Returns from some investment securities follow mean-reverting processes. Presented in this paper are different methods for solving the portfolio optimization problem, with and without serial correlation of returns.

1 Static Portfolio Selection Models

Portfolio selection involves deciding how best to allocate wealth among several risky assets. Static models for portfolio selection restrict the decision making process and the future foresight to a single time period. Portfolio revisions can occur as often as required however each revision is made independently of any other revision. Most static portfolio selection models utilize an expected utility approach. The traditional method for portfolio selection, the Markowitz model, is a static model that uses a quadratic utility function. In practice, most fund managers consider a different static model that maximizes a value-added utility function in terms of a benchmark portfolio. Both methods are described in detail below.

1.1 The Markowitz Model

Markowitz [5] investigated portfolio selection according to the "expected returns-variance of returns" rule. This rule states that when building a portfolio, one wishes to maximise expected return while minimizing the variance (risk) of return. As a consequence, a diversified portfolio is always preferable to a non-diversified portfolio.
The Markowitz model seeks an optimal portfolio that either minimizes risk for a given level of return, or maximizes expected return for a given level of risk. In the mathematical development of his model Markowitz assumes that the returns on each investment are static random variables, independent from period to period, and that the returns of different securities may be correlated at each time step. The Markowitz portfolio selection model can be stated as

\[
\begin{align*}
\text{maximize} & \quad e^T x - \rho x^T V x \\
\text{subject to} & \quad \sum_{j=1}^{N} x_j = 1, \\
& \quad x_j \geq 0 \quad j = 1, \ldots, N.
\end{align*}
\]

where \(e\) is the expected return vector, \(V\) is the variance-covariance matrix for the securities, \(\rho\) is a risk aversion factor, and \(x_j\) gives the proportion of funds invested in security \(j\).

For any given portfolio an expected return, \(E = e^T x\), and a variance, \(V = x^T V x\), can be calculated. Any portfolio with minimum \(V\) for a given \(E\), or maximum \(E\) for a given \(V\) is termed an efficient portfolio by Markowitz. The set of efficient portfolios make up the efficiency frontier, shown in Figure 1. According to Markowitz's "expected returns-variance of returns" rule, an investor would only want to select a portfolio on this efficiency frontier.

1.1.1 Mean-Variance and Utility
Whittle [9] shows how the mean-variance approach can be viewed as an approximation to a utility maximization model. Formally, let \(r\) be the vector of (uncertain) future returns for a set of securities. The problem of portfolio optimization can be thought of as maximizing the utility function

\[
f(x) = E[H(r^T x)]
\]

where \(E\) is the expectation operator, \(x\) is the resulting portfolio and \(H\) is a concave function, subject to constraints on the portfolio (\(\sum x = 1, \quad x \geq 0\)).

Let the expected value and variance of the portfolio return, \(r^T x\), be denoted by \(\mu\) and \(\nu\) respectively. If \(H\) can be expanded in powers of deviation of \(r^T x\), and assuming that powers higher than the second are negligible, then

\[
f(x) = E[H(r^T x)] \sim H(\mu) + \frac{1}{2} \nu H''(\mu)
\]

where the prime indicates differentiation. Using a Taylor's series expansion and ignoring powers higher than the second,

\[
H^{-1}(f(x)) \sim \mu + \frac{H''(\mu)}{2H'(\mu)} \nu.
\]
If $H^{-1}$ is monotone increasing in the effective range of $f(x)$, then the maximization of $f(x)$ is equivalent to the maximization of $H^{-1}(f(x))$ or of

$$f^*(x) = \mu - \frac{1}{2} \theta u$$

(4)

where $\theta$ equals $-H''(\mu)/H'(\mu)$. When $H$ is exponential $\theta$ is independent of $\mu$ and expression (4) is a quadratic function of $x$. Thus the mean-variance model can be thought of as an approximation to the exponential utility function with different levels of absolute risk aversion.

1.1.2 Implementation

The Markowitz model takes as input the expected returns and variance-covariance of the securities that will be in the portfolio. The solution given by the Markowitz model will only be as good as the data that are input. There are many ways to estimate the expected returns and the variance-covariance matrix of the securities. Some examples are included below:

1. Given a time series of past returns for the securities, the variance-covariance matrix can be estimated by the sample variances and covariances from a set of observations taken from the time series, and the sample average return for each security can be used as an estimate of the expected return for that security.

2. Multivariate regression or time series models can be fitted to a time series of past returns. These models can then be used to estimate future expected returns and the variance-covariance matrix.

3. Various methods can be used to transform raw information, such as tips and buy and sell recommendations, into forecasts of expected returns. Refer to Grinold and Kahn [2].

1.1.3 Assumptions and Limitations

The major assumption made in the derivation of the Markowitz portfolio optimization model is that the returns of the securities are independent over time. If this assumption is true then the Markowitz model is statistically correct (see Samuelson [8]). However this assumption is not valid for all securities. Returns from some investment securities have been shown to follow mean-reverting processes. If returns are serially correlated then the Markowitz model is not optimal. Hakansson [4] proves that when returns are serially correlated only the logarithmic utility function is independent of future returns. Thus with time dependent returns, it is not optimal to consider only a one-period horizon using a quadratic utility function. The quadratic utility function of the Markowitz model has also been criticised [3] for the way it implies that risk aversion increases with "wealth".

The Markowitz model is also limited in the way it uses variance as a measure of investment risk. In the calculation of variance positive and negative deviations contribute equally. This means that over performance relative to a mean is penalised
as much as under performance. Because of this the concept of downside risk was
developed. The idea of downside risk is that risk can be measured by the probabil­
ity of the return falling below a specified level or benchmark. This benchmark will
be different for different investors, and is related to the investor’s objectives.

1.2 Portfolio Selection in Practice

In practice, portfolio selection begins with a benchmark portfolio. A benchmark
portfolio is used by the trustee of a fund as a way of conveying to a portfolio man­
ger clear instructions regarding their responsibilities and limitations. For example,
a benchmark for a New Zealand equity manager may be the portfolio of equities
making up the NZSE40 index. The performance of the manager is then judged
relative to the performance of the benchmark. There are two styles of portfolio
management. Passive portfolio management involves trying to replicate the perform­
ance of the benchmark. Active portfolio management involves applying analysis
and process to try and outperform a benchmark.

Assuming the benchmark is an efficient portfolio, consensus forecasts for risk
and return can be calculated. These consensus forecasts will result in an efficiency
frontier (refer to Figure 1) that includes the benchmark portfolio. A passive man­
ger would chose a portfolio on this frontier. Active management starts when the
managers forecasts differ from the consensus. In this case the benchmark portfo­
lio will not necessarily be an efficient portfolio. The portfolio chosen on the new
efficiency frontier will depend on the the manager’s objective function.

The traditional Markowitz approach of using the mean/variance criterion to
choose the portfolio typically leads to portfolios that are too risky for active man­
gagers. Active managers are much more adverse to the risk of deviation from the
benchmark than they are to risk in return of the benchmark portfolio. This is
because the investor bears the risk of the benchmark, whereas the manager bears
the risk of deviating from the benchmark. Instead of looking at total return, active
managers focus on the active component of return and look at active risk/return
tradeoffs. Active return is defined as the difference between the return on the
manager’s portfolio and the benchmark portfolio. Active managers choose a port­
folio that maximizes a value added utility function. Value added is a risk adjusted
expected return that ignores any contribution from the benchmark to risk and
expected return.

1.2.1 Comparison with the Markowitz Model

Like the Markowitz model, active portfolio managers solve the portfolio optimiza­
tion problem by maximising a quadratic utility function in terms of expected re­
turn and variance. The two methods differ in the definition of risk and return. The
Markowitz model considers total risk and total return whereas active managers con­
sider risk and return relative to a benchmark. When comparing the performance
of these two methods one would expect the Markowitz method to outperform the
active management method. This is because of the active management method re­
stricts the set of possible portfolios that can be chosen to be close to the benchmark.
Future work is to be done on performance analysis of these two methods.

Active portfolio management suffers from the same problem as the Markowitz
model when returns are serially correlated.
2 Dynamic Portfolio Selection

An alternative approach to static portfolio selection is to consider the entire planning horizon at the same time. This requires the portfolio optimization problem to be reformulated in terms of a multi-period framework. The dynamic nature of the portfolio selection problem, for example the fluctuating security returns over time, can then be modelled as a multi-stage stochastic program. As well as capturing the dynamic nature of the problem, this also allows for a number of realizations for the uncertain quantities. Stochastic programming models for the portfolio selection problem are discussed in detail below.

2.1 Stochastic Programming for Portfolio Optimization

Dynamic portfolio selection problems can be modelled as either stochastic network models [7] or multi-stage stochastic programs [1].

Stochastic network models formulate the problem as a time-expanded network flow problem. A network model is constructed by associating a network node with each security-time period pair, and an arc with each transaction decision. Each arc may have an associated multiplier that can be used to represent, for example, exchange, return or borrowing rates. External deposits or withdrawals correspond to supplies and demands at the nodes, and flow conservation constraints are present at the nodes.

Multi-stage stochastic programming models formulate the problem as a linear (or non-linear) program with a dynamic matrix structure. The matrix appears as a staircase pattern of blocks, with each block corresponding to a different time period.

Uncertainty in both models is represented by a number of distinct realizations. Each complete realization of all unknown parameters is known as a scenario. As the scenarios and their corresponding probabilities are major inputs into the stochastic models, it is important that the scenario set is representative. A number of authors discuss scenario generation techniques (for example, see Mulvey and Vladimirou [6]).

The aim of both methods is to generate an investment recommendation for the current period that does not depend on previous events, but takes into account all postulated scenarios and their respective probabilities. The goal is to maximise the expected utility of the value of the portfolio at the end of the planning horizon.

2.1.1 Comparison with Static Portfolio Optimization

Multi-stage stochastic models for portfolio optimization have a number of advantages over static portfolio optimization. Stochastic models allow for serial correlation of the returns. The serial correlation is incorporated into the model via the return scenarios. Future liabilities, anticipated external deposit/withdrawal cash streams and liquidity requirements can also be considered in a stochastic multi-stage model.

However, if the security returns are independent over time then the Markowitz model has an advantage over multi-stage stochastic models. The scenarios that are used in the stochastic models are (usually) generated from a variance-covariance matrix of the security returns. The variance-covariance of the resulting scenario
set thus provides only an approximation to the original variance-covariance matrix. The Markowitz model uses the actual variance-covariance matrix of returns. As static models are statistically correct when returns are independent, the Markowitz model therefore has an added advantage.

Future work is to be done on performance analysis of the Markowitz model and stochastic multi-stage programming models, with and without serial correlation of yields.

Acknowledgements

I would like to thank the Foundation for Research, Science and Technology for their generous support. I would also like to thank Tower Portfolio Management for supplying the necessary data for my experiments, and Associate Professor Andy Philpott for providing excellent supervision of my work.

References


