Solution and Application of Nonconvex Network Flow Problems

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Abstract

Nonconvex network flow models have numerous applications in a variety of fields, including network design, facility location, production planning, physical distribution, electricity transmission, and telecommunication problems. A variety of methods for solving nonconvex networks have been developed in the literature: however, global optimisation of this class of problem is complex, and the convergence of the procedures slow. In this paper we discuss a technique called "enhanced capacity improvement" that can be used in conjunction with standard solution procedures. When used in conjunction with a branch-and-bound algorithm for a series of nonconvex network flow problems, enhanced capacity improvement was able to find the global optimal solutions from two to sixty times faster than the traditional approach.

1. Introduction

In this paper, we examine solution methods for network flow problems in which the cost functions are arbitrary nonlinear functions. Such network optimisation problems arise in a variety of contexts involving discounting or other economies of scale. Examples include multicommodity multiattribute network flow problems [13], facility planning in telecommunication networks [1], and natural gas pipeline systems [7]. Other application areas include production and inventory planning, transportation network design, facility location, hydroelectric scheduling, electric transmission network operation and expansion, and air traffic control [5,6].

In cases where the network cost functions are linear or convex for each arc, very efficient solution methods exists (see surveys in [11,15]). However, there are many cases in which the arc costs must be expressed as nonconvex functions. It is possible to convert problems involving arbitrary nonconvex functions into problems involving either a concave or convex objective function on each arc [10]. However, determining the globally optimal solution to such problems is challenging, because a local optimum is not necessarily a global one; and the number of locally optimal points can be enormous, even for moderate-sized problems.

Deterministic solution procedures for such problems include vertex ranking [12], branch-and-bound [4, 8], dynamic programming [3], and decomposition [16]. However, convergence is often slow, and thus the performance of these approaches may not be satisfactory. One attempt to overcome this shortcoming is by the use of *capacity improvement*. Used in conjunction with a global network optimisation algorithm, most typically branch-and-bound, capacity improvement aims to decrease the solution time by effective contraction of the feasible region. Capacity improvement based on a *linear* relaxation of the minimum cost network flow problem has been studied extensively in the literature (see, for example, [9,14]). We extend these results by using a *nonconvex* relaxation of the problem. The range reduction produced by the nonconvex relaxation is considerably greater than that produced by the linear relaxation.

The paper is organised as follows. First, Section 2 formulates the general minimum cost network flow problem. Section 3 then develops the enhanced capacity improvement procedure. Results from computational testing of capacity improvement in the context of a branch and bound algorithm are presented in Section 4. This section shows that a 50% to 98% reduction in CPU time can be achieved using the enhanced capacity improvement technique. Finally, Section 5 summarises the paper and discusses directions for future research.

2. Problem Formulation

Mathematically, we can describe minimum cost network flow problems with arbitrary arc cost functions as follows:

$$(Q) \qquad \textit{global min } \Phi(\underline{x}) \text{ s.t. } \underline{x} \in X = G \cap H$$

where I is the node set with generic element i and cardinality m, J is the (directed) arc set with generic element j and cardinality n, $\underline{x} = \{x_1, \ldots, x_j, \ldots, x_n\}$ is the vector of decision variables, $\Phi(\underline{x}) = \sum_{j \in J} \phi_j(x_j)$ is the objective function for the problem, and $X = G \cap H$ is the set of feasible values for $\underline{x}$. G is the set of conservation of flow constraints

$$G = \left\{ \underline{x}: \sum_{j \in J} g_{ij} x_j - b_i = 0 \ \ \forall\, i \in I \right\} \tag{1}$$

where $g_{ij} \in \{-1,0,1\}$, $\sum_{i \in I} g_{ij} = 0$ and $\sum_{i \in I} |g_{ij}| = 2 \ \ \forall\, j \in J$, and b_i is the net supply (if b_i is positive) or demand (if b_i is negative) for node i (note that $\sum_{i \in I} b_i = 0$). The hyperrectangle $H \subset \Re^n$ is given by

$$H = \{\underline{x}: \underline{l} \le \underline{x} \le \underline{u}\} \tag{2}$$

where $\underline{l} = \{l_1, \ldots, l_j, \ldots, l_n\}$ and $\underline{u} = \{u_1, \ldots, u_j, \ldots, u_n\}$ are the lower and upper flow bounds on the decision variable vector $\underline{x}$. We also define x_j^* as the value of x_j in the optimal solution to problem Q, $v[Q]$ as the optimal objective value for problem Q, and $ub[Q]$ as an upper bound for $v[Q]$.

3. Capacity Improvement

In this section we develop the nonconvex enhanced capacity improvement procedure. Capacity improvement reduces the size of the feasible region by systematically developing tighter lower and upper arc flow bounds. We first define two relaxations of subproblem Q. The first, $\overline{Q}$, is defined as

$(\overline{Q})$ $$min\ \overline{\Phi}(\underline{x})\ s.t.\ \underline{x} \in X = G \cap H$$

where $\overline{\Phi}(\underline{x}) = \sum_{j \in J} \overline{\phi}_j(x_j)$ is an affine lower envelope of $\Phi(\underline{x})$ over the hyperplane H. It is important to note that problem $\overline{Q}$ is a pure network flow problem, and thus can be readily solved. To define the second relaxation, $\hat{Q}$, we first partition set J into $J = B \cup NL \cup NU$, where $j \in B$ if arc j is basic, $j \in NL$ if arc j is nonbasic at its lower bound, and $j \in NU$ if arc j is nonbasic at its upper bound in the optimal solution to $\overline{Q}$. Relaxation $\hat{Q}$ is defined as

$(\hat{Q})$ $$min\ \hat{\Phi}(\underline{x})\ s.t.\ \underline{x} \in \hat{X} = G \cap \hat{H}$$

where $\hat{\Phi}(\underline{x}) = \sum_{j \in J} \hat{\phi}_j(x_j)$ is a nonconvex relaxation of $\Phi(\underline{x})$ over H, and $\hat{H} = \{\underline{x} : \hat{\underline{l}} \le \underline{x} \le \hat{\underline{u}}\}$, with $\hat{\underline{l}} = \{\hat{l}_1, \ldots, \hat{l}_j, \ldots, \hat{l}_n\}$ and $\hat{\underline{u}} = \{\hat{u}_1, \ldots, \hat{u}_j, \ldots, \hat{u}_n\}$ given by

$$\hat{l}_j = \begin{cases} l_j & if\ j \in NL \\ -\infty & if\ j \in NU \cup B \end{cases} \text{ and } \hat{u}_j = \begin{cases} u_j & if\ j \in NU \\ +\infty & if\ j \in NL \cup B \end{cases} \tag{3}$$

We also let $\hat{x}_j$ denote the value of the decision variable x_j in the optimal solution to problem $\hat{Q}$, and $v[\hat{Q}]$ denote the optimal objective function value of problem $\hat{Q}$.

To define $\hat{\phi}_j(x_j)$ for each arc $j \in J$, if $j \in NL \cup NU$ and $\phi_j(x_j)$ is concave, we define a scalar η_j for each arc $j \in J$ as being equal to l_j if $j \in NU$ and u_j if $j \in NL$. We then define

$$\hat{\phi}_j(x_j) = \begin{cases} \min\{\phi_j(x_j), \phi_j(\eta_j) + \bar{z}_j \cdot (x_j - \eta_j)\} & if\ l_j \le x_j \le u_j \\ \overline{\phi}_j(x_j) & otherwise \end{cases} \tag{4}$$

where $\bar{z}_j = \sum \bar{\pi}_i \cdot g_{ij}$ and $\bar{\pi}_i$ is the dual variable associated with node i in the optimal solution to problem $\overline{Q}$. Otherwise, we simply set $\hat{\phi}_j(x_j) = \overline{\phi}_j(x_j)$. Figure 1 shows a typical representation of $\hat{\phi}_j(x_j)$ when $j \in NL$ and $\phi_j(x_j)$ is concave.

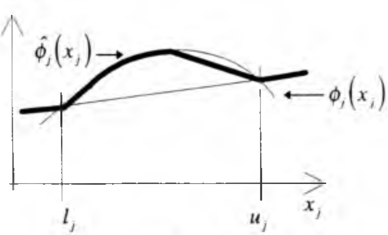

Figure 1. Typical arc cost function $\hat{\phi}_j(x_j)$ when $j \in NL$ and $\phi_j(x_j)$ is concave

Problem $\hat{Q}$ has been constructed so that the optimal solution to $\overline{Q}$ is also optimal to $\hat{Q}$. Parametric analysis of problem $\hat{Q}$ is also especially easy to perform. In particular, we may examine how the solution to $\hat{Q}$ changes as the value of a single arc varies from its optimal solution value in problem $\hat{Q}$. To aid the parametric analysis of $\hat{Q}$, we define a *reduced cost function* for each arc $j \in J$, denoted $\Delta_j(x_j)$, as follows:

$$\Delta_j(x_j) = \hat{\phi}_j(x_j) - \bar{z}_j x_j \tag{5}$$

The reduced cost function is a straight forward extension of the reduced cost coefficient used in linear programming. We now consider varying a *single* arc $k \in J$ by a scalar amount δ_k from its value in the optimal solution to $\hat{Q}$. If $k \in NL$, the increase in the objective function of problem $\hat{Q}$ when arc *k* is varied from l_k is given by

$$\theta_k(\delta_k) = \begin{cases} +\infty & \text{if } \delta_k < 0 \\ \Delta_k(l_k + \delta_k) - \Delta_k(l_k) & \text{if } \delta_k \geq 0 \end{cases} \tag{6}$$

The parametric function $\theta_k(\delta_k)$ can be similarly calculated when $k \in NU$ and $k \in B$.

The capacity improvement calculations use the parametric analysis of problem $\hat{Q}$ discussed above. Specifically, for arc *k* we define two values of δ_k, denoted δ_k^- and δ_k^+, as follows:

$$\delta_k^- = \min\{\delta_k : \theta_k(\delta_k) \leq ub[Q] - v[\hat{Q}]\} \tag{7}$$

$$\delta_k^+ = \max\{\delta_k : \theta_k(\delta_k) \leq ub[Q] - v[\hat{Q}]\} \tag{8}$$

Recall that $\hat{Q}$ is a relaxation of *Q*, and therefore $\hat{x}_k + \delta_k^-$ forms a lower bound, and $\hat{x}_k + \delta_k^+$ an upper bound, to x_k^*. We use these results to define new lower and upper bounds to x_k^* as follows:

$$l_k^{new} = \max\{l_k, \hat{x}_k + \delta_k^-\} \tag{9}$$

$$u_k^{new} = \min\{u_k, \hat{x}_k + \delta_k^+\} \tag{10}$$

New upper and lower flow bounds can be calculated in this way for all arcs $j \in J$ that are at least as "tight" as the original bounds. When used in as part of a global optimisation procedure, for example branch-and-bound, the capacity improvement procedure works by decreasing the size of the solution space and thus enabling tighter relaxations of problem *Q* to be calculated. Figure 2 shows the calculation of l_k^{new} and u_k^{new} for a typical arc $k \in B$, and the resulting tighter affine relaxation of $\phi_j(x_j)$.

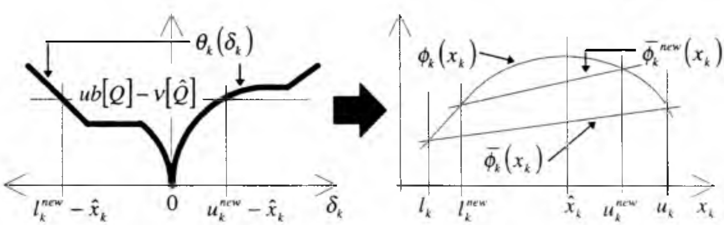

Figure 2. Capacity improvement bound calculation for a typical arc $k \in B$

4. Computational Analysis

The enhanced capacity improvement procedure discussed in the previous section was implemented in conjunction with a branch-and-bound procedure and applied to two groups of minimum cost network flow problems. The first group (labelled QTP-A1 through QTP-C5) was a set of 15 *transportation* problems, each consisting of 12 supply nodes and 8 demand nodes. The second group (labelled QTS-A1 through QTS-C5) was a set of 15 *transshipment* problems, each consisting of 5 supply nodes, 4 demand nodes, and 9 transshipment nodes. The 15 test problems in each of the two groups were divided into three subsets denoted by the suffixes -A*, -B*, and -C*. All test problems had concave quadratic arc cost functions of the form $\phi_j(x_j) = \alpha_j x_j + \beta_j x_j^2$ where the coefficients α_j were randomly chosen from a uniform distribution in the range [0,500], and β_j was randomly chosen from a uniform distribution in the range [-20,0] for the -A* problems, [-500,0] for the -B* problems, and [-10000,0] for the -C* problems.

Each test problem was solved twice: first, using a standard branch-and-bound procedure for minimum concave cost network flow problems (see, for example, [5]); and second, using branch-and-bound with enhanced capacity improvement. Both solution procedures were programmed in the C language using the CPLEX [2] callable library. All problems were solved on an IBM compatible 486 DX-100 personal computer. All test problems were solved to optimality, and the total CPU time recorded.

		CPU Time (seconds)		Improvement
Group	Name	Standard B&B	Enhanced C.I.	Ratio
Transportation	QTP-A1 - QTP-A5	65.6	29.0	2.3
	QTP-B1 - QTP-B5	161.2	69.8	2.3
	QTP-C1 - QTP-C5	250.2	132.5	1.9
Transshipment	QTS-A1 - QTS-A5	2142.2	33.3	64.3
	QTS-B1 - QTS-B5	1172.1	83.4	14.1
	QTS-C1 - QTS-C5	1068.6	33.2	32.2

Table 1. CPU time

Table 1 contains the average CPU time (in seconds) for each of the six subsets of five subproblems. From these results, it is clear enhanced capacity improvement provided a substantial decrease in the solution times over the traditional branch-and-bound procedure for both groups of test problems. Enhanced capacity improvement performed particularly well on the more complex transshipment problems, decreasing the average solution time by a ratio of 15 to 60 times.

5. Conclusions

This paper has examined the application and solution of minimum cost network flow problems with nonconvex arc cost functions. In particular, an extension to the standard solution methods for these problems, enhanced capacity improvement, was presented. When used in conjunction with a branch-and-bound procedure, enhanced capacity improvement improved solution times for a set of test problems by a substantial degree over the branch-and-bound procedure alone. This was due to an improvement in the lower and upper flow bounds producing a correspondingly tighter convex envelope of the objective function. This produced tighter subproblem relaxations in the branch-and-bound procedure, which enabled the subproblems to be fathomed more efficiently.

Future research will encompass three areas. First, the capacity improvement theory will be extended to optimisation problems involving feasible regions more general than the network flow constraints considered in this paper. Second, nonconvex relaxations other than the one presented in this paper will be developed. Such relaxations will aim to provide even tighter arc flow bounds which, in turn, can reduce solution times still further. Finally, capacity improvement methods will be applied to the problem of electricty transmission network expansion.

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