On Dominated Terms in the General Knapsack Problem

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Abstract
A necessary and sufficient condition for the identification of dominated terms in a general knapsack problem is derived. By general we mean a knapsack problem which is unbounded, equality constrained and which has a parametric right hand side. The given condition yields published results in the literature. A report on computational experiments for large scale knapsack problems, demonstrating the effectiveness of this approach, is included.

1 Introduction

A knapsack problem (KP) is a NP-hard and basic problem in Integer Programming (IP) (see Martello and Toth [3]). We have

Definition 1. The general knapsack problem (called KP(1)) is defined in the form:

\[ F(b) = \max \sum_{j \in I} c_j x_j \]

subject to
\[ \sum_{j \in I} a_j x_j = b, \quad (1.1) \]
\[ x_j \geq 0 \text{ integer, } \forall j \in I, \]

where \{a_j\} are given positive integers, \{c_j\} are given arbitrary integers, b is a non-negative integer parameter, \( j \in I = \{1, 2, ..., n\} \) with \( n \geq 2 \).

If \{c_j\} (\( j \in I \setminus \{n\} \)) are given positive integers, \( c_n = 0 \) and \( a_n = 1 \), then the problem is an ordinary KP (called KP(2)). By a given KP we mean the KP with a fixed \( b = b_0 \). For a given KP(2), some results and algorithms can be seen in [3]. For a given KP(1), an algorithm has been obtained by Yanasse and Soma [6]. It
is a significant work that a lot of redundant terms of the problem are eliminated prior to starting a solution method. For KP(2), an identifying criterion and an eliminating procedure on dominated item types are given by [3]. Further work on dominance relations for KP(2) has been done by Dudziński [2], and by Pisinger [5]. A similar result to [3] is provided by Babayev and Mardanov [1], and that result is extended for the multidimensional KP and Mixed IP problems in [1].

In this paper, we are interested in discussing how to identify and eliminate dominated terms in KP. A necessary and sufficient condition on identification of dominated terms is provided. Several identifying criteria are set up, and Theorem 3.2 in [3] can be considered as a special case of our basic result. Computational experiments for large scale problems are reported.

2 The Main Results

The KP(1) is in general harder than KP(2) since in KP(1) there may not exist any feasible solution for some values of $b$. For a given KP(1), we call term with index $k \in N$ (term $k$ for short) the redundant term if the optimal solution value $F(b_0)$ (if it exists) does not change when $k$ is removed from $N$. For a redundant term $k$, we can fix $x_k = 0$ in the original problem. In KP(1), there may exist some terms that can be removed for all nonnegative integer values of the parameter $b$ without altering the optimal solution value $F(b)$ (if it exists). Our aim is to identify and try to eliminate this kind of the redundant term. We first give two definitions and a basic result.

**Definition 2.** A positive integer $a_k$ has a representation by the positive integers $\{a_j\} (j \in N \setminus \{k\})$ if there exist nonnegative integers $\{l_j\}$, such that $a_k = \sum_{j \in N \setminus \{k\}} l_j a_j$.

**Definition 3.** In KP(1), term $k$ is dominated if the optimal solution value $F(b)$ (if it exists) does not change when $k$ is removed from $N$.

**Theorem 1.** In KP(1), term $k$ is dominated if and only if there exist nonnegative integers $\{l_j\}$, such that

$$a_k = \sum_{j \in N \setminus \{k\}} l_j a_j, \quad c_k \leq \sum_{j \in N \setminus \{k\}} l_j c_j.$$  \hfill (2.1)

**Proof.** ($\Rightarrow$) If a feasible solution to KP(1) has the form: $x_j = h_j \ (\forall j \in N \setminus \{k\})$, and $x_k = h_k \geq 1$, then there exists a new feasible solution that has the form: $x_j = h_j + l_j h_k \ (\forall j \in N \setminus \{k\})$, and $x_k = 0$. We have

$$\sum_{j \in N} c_j h_i \leq \sum_{j \in N \setminus \{k\}} c_j h_j + h_k \cdot \sum_{j \in N \setminus \{k\}} c_j l_j = \sum_{j \in N \setminus \{k\}} c_j (h_j + l_j h_k) + c_k 0.$$
Thus the new solution is at least as good as the old one.

(⇒) Let term \( k \) be dominated, and \( k \) be eliminated from \( N \) in KP(1). Suppose \( a_k \) has no any representation by other \( \{a_j\} \). When \( b = a_k \), \( F(a_k) \) does not exist in the new KP(1). However in the original KP, \( F(a_k) = c_k \), this contradicts Definition 3. Therefore, \( a_k \) has at least one representation by other \( \{a_j\} \). Now suppose that for this term \( k \) it is the case that if \( a_k = \sum_{j \in N \setminus \{k\}} l_j a_j \) with integers \( l_j \geq 0 \) (\( \forall j \in N \setminus \{k\} \)), it is always true that \( c_k > \sum_{j \in N \setminus \{k\}} l_j c_j \). When \( b = a_k \), in the new KP

\[
F(a_k) = \max \sum_{j \in N \setminus \{k\}} l_j c_j \geq \sum_{j \in N \setminus \{k\}} l_j^* c_j.
\]

However in the original KP, \( F(a_k) = c_k > \sum_{j \in N \setminus \{k\}} l_j^* c_j \), so we arrive at a contradiction. Therefore, condition (2.1) is satisfied. □

The significance of Theorem 1 is that all of the dominated terms in KP(1) are identified. Below, several concrete identifying criteria are set up based on Theorem 1 to try to eliminate these dominated terms. We first have

**Theorem 2.** In KP(2), if there are two distinct terms \( j \) and \( k \) \((j, k \in N \setminus \{n\})\) such that

\[
[a_k/a_j]c_j \geq c_k,
\]

then term \( k \) is dominated.

**Proof.** In KP(2), \( a_n = 1 \) and \( c_n = 0 \). There always exists a nonnegative integer \( r_j < a_j \) satisfying \( a_k = [a_k/a_j]a_j + r_j a_n \). By condition (2.2), \( c_k \leq [a_k/a_j]c_j + r_j c_n \). Thus term \( k \) is dominated by Theorem 1 \((l_j = [a_k/a_j], l_n = r_j, \text{ and } l_i = 0 \text{ otherwise})\). □

Since in KP(2) \( a_n = 1 \) and \( c_n = 0 \), using Theorem 1 it is also easy to obtain the following property: **term \( n \) is dominated in KP(2) if and only if there exists at least one \( j \in N \setminus \{n\} \) such that \( a_j = 1 \).

Theorem 2 above can be regarded as Theorem 3.2 in [3]. The \( O(n^2) \) eliminating procedure for dominated terms satisfying condition (2.2) (see Corollary 3.1 in [3]) is an important part of the algorithm of [3] for solving a large scale KP(2). In [2], it is shown that the dominance relation (2.2) is a partial order leading to more efficient eliminating procedures for KP(2). In [5], using condition (2.2) (Definition 2 in [5]) a faster reduction method for KP(2) is discussed by sorting according to nondecreasing weights \( \{a_j\}_{j \in N \setminus \{n\}} \). Note that by Theorem 1, \( a_n = 1 \) and \( c_n = 0 \), it is easy to yield an identifying criterion (Definition 1 in [5]) for KP(2) given by [5].

In addition, Theorem 2 can not be directly used or generalized to eliminate general dominated terms in KP(1). Although the constraint equation in KP(1) can be equivalently transformed into two inequalities:

\[
\sum_{j \in N} a_j x_j \leq b, \quad -\sum_{j \in N} a_j x_j \leq -b,
\]
using Theorem 3 in [1], in which Theorem 3 is an extended result from Theorem 1
of [1] (this Theorem 1 is similar to Theorem 3.2 in [3] or Theorem 2 above), only
term $k$ that satisfies $[a_k/a_j]c_j \geq c_k$ and $[a_k/a_j] = a_k/a_j$ (i.e., $a_k = l_ja_j$, where
integer $l_j > 0$) can be identified as a dominated term in KP(1), where $j,k \in N$, and
$j \neq k$. In fact, this kind of dominated term is easy to be identified by our Theorem
1. Using Theorem 1, a new criterion for KP(2) is derived:

**Theorem 3.** In KP(2), if there are three distinct terms $i$, $j$ and $k$ ($i,j,k \in
N \{n\}$) such that

$$
\left( a_k - \left[ a_k/a_j \right] a_j \right)/a_i c_i + \left[ a_k/a_j \right] c_j \geq c_k,
$$

then term $k$ is dominated.

**Proof.** From the Proof of Theorem 2 we know $a_k = \left[ a_k/a_j \right] a_j + r_j$, where
$0 \leq r_j = a_k - \left[ a_k/a_j \right] a_j$. There always exists an integer $r_i$ satisfying $0 < r_i < a_i$
such that $r_j = \left[ r_j/a_i \right] a_i + r_i a_n$. Hence $a_k = \left[ a_k/a_j \right] a_j + \left[ r_j/a_i \right] a_i + r_i a_n$. Letting
$l_j = \left[ a_k/a_j \right]$, $l_i = \left[ r_j/a_i \right]$, $l_n = r_i$ and $l_h = 0$ otherwise, by condition (2.3) and
Theorem 1, term $k$ is dominated.

From Theorem 3, a corresponding $O(m^2)$ eliminating procedure is easily ob­
tained, where $m$ is the number of variables of an original KP(2) or an equivalent
 KP(2) that is obtained by first applying Theorem 2 to the original KP(2). Theorem
3 can be further developed into the criterion including more terms.

In the following we give some identifying results for KP(1). First we provide the
formulas which can be proved by the induction method:

**General Inequalities.** Let $\{c_i\}$ be real, $\{a_i\}$ be positive, $\{l_i\}$ be nonnegative
with at least one $l_i > 0$, $i \in \{1, \ldots, m\}$ with $m \geq 2$. Then the following inequalities
hold:

$$
\min \left\{ \sum_{1 \leq i \leq m} l_i c_i \right\} \leq \sum_{1 \leq i \leq m} l_i a_i \leq \max \left\{ \sum_{1 \leq i \leq m} l_i c_i \right\}.
$$

In KP(1) assume that

$$
c_i/a_1 = \max_{j \in N} \{c_j/a_j\},
$$

and if $c_1/a_1 = c_i/a_i$ then $a_1 \leq a_i$ ($i \in N \{1\}$). This kind of KP(1) is called
KP(1'). When $a_1 = 1$, KP(1') obviously has the optimal solution: $x_1^* = b$, and
$x_j^* = 0$ otherwise with $F(b) = c_1 b$. Therefore, we always assume $a_1 > 1$ in KP(1').
If there exists $k \in N \{1\}$ such that $c_1/a_1 = c_k/a_k$ and $a_1 = a_k$, it is reasonable to
always assume term $k$ is dominated and is eliminated in KP(1'). Using (2.4) we
have proved the following property:

**A Property of KP(1').** In KP(1') in which if $c_1/a_1 = c_i/a_i$ then $a_1 < a_i$ is set
for $i \in N \{1\}$, term 1 is undominated.
Theorem 4. In KP(1), if there are two distinct terms \( j \) and \( k \) \((j, k \in N \setminus \{1\})\) such that

\[
a_j \equiv a_k \pmod{a_i}, \quad a_j \leq a_k, \quad \text{and} \quad p_j \leq p_k; \quad (2.6)
\]

or

\[
a_k \equiv 0 \pmod{a_i}, \quad (2.7)
\]

where \( a_i (> 1) \) satisfies \( c_1/a_1 = \max_{j \in N} \{c_j/a_j\} \), and \( p_i \triangleq c_1a_i - c_ia_1 \ (\forall i \in N \setminus \{1\}) \), then term \( k \) is dominated.

Proof. By condition (2.6), we have \( a_k = l_1a_1 + a_j \), where integer \( l_1 \geq 0 \). Since \( p_j \leq p_k \), we have \( c_1a_j - c_ja_1 \leq c_1a_k - c_ka_1 = c_1(l_1a_1 + a_j) - c_ka_1 \). Thus \( c_k \leq l_1c_1 + c_j \).

Hence term \( k \) is dominated using Theorem 1 \((l_j = 1)\). By condition (2.7), we have \( a_k = l_1a_1 \), where integer \( l_1 > 0 \). Since \( c_1/a_1 \geq c_k/a_k = c_k/l_1a_1 \), \( c_k \leq l_1c_1 \). Hence term \( k \) is dominated. □

The use of a congruence modulo \( a_i \) is linked with solving the group knapsack problem (GKP), which is a relaxation problem of KP, in which \( p_i \) can be considered as the coefficient of \( x_i \) in the objective function of the GKP. See for example, Nemhauser and Wolsey [4] for KP(2), and Zhu [7] for KP(1). Using Theorem 4, an \( O(n^2) \) eliminating procedure for KP(1) is easily devised. Note that if using KP(1') in place of KP(1) in the Theorem, more dominated terms may be eliminated since term 1 in KP(1') always is an undominated term. In addition, Theorem 4 can be applied to the KP(2), and can easily be extended to the following criterion:

Corollary. In KP(1), if there are two distinct terms \( j \) and \( k \) \((j, k \in N \setminus \{\tau\})\) such that

\[
ca_j \equiv a_k \pmod{a_\tau}, \quad ca_j \leq a_k, \quad \text{and} \quad cq_j \leq q_k; \quad (2.8)
\]

or

\[
a_k \equiv 0 \pmod{a_\tau}, \quad \text{and} \quad 0 \leq q_k, \quad (2.9)
\]

where \( c \) is a given positive integer, given \( \tau \in N \) such that \( a_\tau > 1 \), and \( q_i \triangleq c_\tau a_i - c_ia_\tau \ (\forall i \in N \setminus \{\tau\}) \), then term \( k \) is dominated.

The Proof of the Corollary is similar to the Proof of Theorem 4, so is omitted. Like Theorem 3 the Corollary is easily developed into the criterion including more terms of KP(1).

Example 1. Consider a KP(2):

\[
F(b) = \max \ 11x_1 + 22x_2 + 6x_3 + 15x_4 + 35x_5 + 3x_6
\]

subject to \[ 4x_1 + 9x_2 + 3x_3 + 6x_4 + 15x_5 + 2x_6 + x_7 = b, \]

\[ x_j \geq 0 \text{ integer, } \forall j \in N, \]

where \( c_1/a_1 > c_j/a_j \ (\forall j \in N \setminus \{1\}) \), and \( b \) is a nonnegative integer parameter.

Using Theorem 3 or 4, it follows that terms 2 and 5 are dominated. Thus terms 2 and 5 can be eliminated from the problem before using a solution method. If we apply Theorem 2 to this example, only term 2 is identified as being dominated.
Example 2. Consider a KP(1):

\[ F(b) = \max 49a_1 + 11x_2 + 22x_3 + 6x_4 + 15x_5 + 35x_6 + 3x_7 \]

subject to \[ 16x_1 + 4x_2 + 9x_3 + 3x_4 + 6x_5 + 15x_6 + 2x_7 = b, \]
\[ x_j \geq 0 \text{ integer, } \forall j \in N, \]

where \( c_1/a_1 > c_j/a_j \) (\( \forall j \in N\setminus\{1\} \)), and \( b \) is a nonnegative integer parameter.

In the equality constrained problem (\( F(1) \) does not exist), Theorem 2 and 3 cannot be applied, and Theorem 4 is invalid. Letting \( c = 1 \) and choosing \( r = 2 \) (or 3 or 5), term 6 is identified being dominated using the above Corollary.

Note that before using the eliminating result for a given KP(1), it does not require to know whether \( F(b_0) \) exists since the original problem is equivalent to the new problem obtained by using the eliminating result.

In the following, we add a condition to KP(1) by assuming

\[ c_1/a_1 \geq c_2/a_2 \geq \ldots \geq c_n/a_n. \] (2.10)

This kind of KP(1) is called \( KP(1^{\prime}) \). Using Theorem 1 and formulas (2.4), we have

**Theorem 5.** In \( KP(1^{\prime}) \), if \( a_k (k \in N\setminus\{1\}) \) has a representation by \( a_1, \ldots, a_{k-1} \), then term \( k \) is dominated.

**Proof.** Let \( a_k = \sum_{1 \leq j \leq k-1} l_j a_j \), where \( \{l_j\} \) are nonnegative integers. We have

\[ \frac{c_k}{a_k} = \frac{c_{k-1}}{a_{k-1}} = \min_{1 \leq j \leq k-1} \left( \frac{c_j}{a_j} \right) \leq \frac{\sum_{1 \leq j \leq k-1} l_j c_j}{\sum_{1 \leq j \leq k-1} l_j a_j} = \frac{\sum_{1 \leq j \leq k-1} l_j c_j}{a_k}. \]

Hence we have \( c_k \leq \sum_{1 \leq j \leq k-1} l_j c_j \). By Theorem 1, term \( k \) is dominated. \( \square \)

Theorem 1 is easily generalized for a mixed integer KP.

**Theorem 6.** In a mixed integer KP, where the form looks like KP(1) but given \( a_j > 0 \), given \( c_j \) real, (\( \forall j \in N \)), parameter \( b \geq 0 \), integer \( x_j \geq 0 \) (\( \forall j_1 \in N_1 \triangleq \{1,...,p\} \)), and real \( x_{j_2} \geq 0 \) (\( \forall j_2 \in N_2 \triangleq \{p+1,...,n\} \)), term \( k \in N_1 \) is dominated if and only if there exist integer \( l_{j_1} \geq 0 \) (\( \forall j_1 \in N_1\setminus\{k\} \)), and real \( l_{j_2} \geq 0 \) (\( \forall j_2 \in N_2 \)), such that

\[ a_k = \sum_{j \in N\setminus\{k\}} l_j a_j, \quad \text{and} \quad c_k \leq \sum_{j \in N\setminus\{k\}} l_j c_j. \] (2.11)

The Proof of the Theorem is similar to the Proof of the Theorem 1, so is omitted. Based on Theorem 6, the effective identifying criteria and efficient computational procedures can be developed. Note that in the special case \( p = n \), Theorem 6 can be considered as Theorem 1. If \( p = 0 \), it is easy to prove by \( a_j = (a_j/a_1)a_1 \) and \( c_j \leq \)
\((a_j/a_i)c_i, \forall j \in N\setminus\{1\}\), that except for term 1, (assuming \(c_1/a_1 = \max_{j \in N}\{c_j/a_j\}\)), all terms can be regarded as being dominated. Therefore, this kind of KP, that is a simple Linear Programming problem, has the optimal solution: \(x_i^* = b/a_1\), and \(x_j^* = 0\) otherwise with \(F(b) = c_1b/a_1\).

### 3 Computational Experiments

In this Section we analyze the experimental results by only using Theorem 2 (TH2) and Theorem 4 (TH4) of this paper to identify dominated terms of KP. Three types of test problems are discussed: first is KP(1), second KP(2), and finally the (unbounded) subset-sum problem (SSP) sometimes called the value independent problem, see [3] or [4]. A SSP can be considered as a completely correlated KP(2): \(c_j = a_j\), where \(a_i < a_j\) is assumed, \(\forall j \in N\setminus\{n\}\).

For KP(1), we consider the data set \(\{(c_j, a_j)\}_{j \in N}\) that the data satisfies condition (2.5), where original \(c_j\) uniformly random in \([1, 1000]\), and original \(a_j\) uniformly random in \([10, 1000]\).

For KP(2), we consider the data set \(\{(c_j, a_j)\}_{j \in N}\cup\{(0, 1)\}_{j=n+1}\), where the set \(\{(c_j, a_j)\}_{j \in N}\) is the same in the above KP(1).

For SSP, we consider the data set \(\{(a'_j, a'_j)\}_{j \in N}\cup\{(0, 1)\}_{j=n+1}\), where we let \(a'_i = a_i = a_1, a'_j = a_j, (\forall j \in N\setminus\{1, i\}\) and \(\{a_j\}_{j \in N}\) comes from the data set \(\{(c_j, a_j)\}_{j \in N}\) of the above KP(1).

The results are presented in the Table below. For each type of testing problem and value of \(n\), the Table reports the mean number of possibly undominated terms (that include in term \(n + 1\) for KP(2) and SSP), computed over 10 problems using TH2 and TH4 respectively. Note that according to the chosen data \((a_j \in [10, 1000], \forall j \in N)\) and discussion in Section 2, term \(n + 1\) \((c_{n+1} = 0, \text{ and } a_{n+1} = 1)\) always is an undominated term for KP(2) and SSP. Thus from theoretical point of view (i.e., analyzing by applying the trivial criterion: in a KP if \(a_j = a_k\), and \(c_j \geq c_k, j \neq k\), then term \(k\) is dominated), except term \(n + 1\) there are at most \(\min\{n, 991\}\) undominated terms in every testing problem.

<table>
<thead>
<tr>
<th></th>
<th>TH2</th>
<th>TH4</th>
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<tbody>
<tr>
<td>(n)</td>
<td>KP(1)</td>
<td>KP(2)</td>
</tr>
<tr>
<td>500</td>
<td>-</td>
<td>3.1</td>
</tr>
<tr>
<td>1000</td>
<td>-</td>
<td>4.0</td>
</tr>
<tr>
<td>2000</td>
<td>-</td>
<td>4.0</td>
</tr>
<tr>
<td>5000</td>
<td>-</td>
<td>3.9</td>
</tr>
<tr>
<td>10000</td>
<td>-</td>
<td>4.6</td>
</tr>
</tbody>
</table>
The computational results show that the number of possibly undominated terms in testing problems after using TH2 or TH4 is very small, and the identifying effect of the Theorems is different to a same type problem.

TH2 works for KP(2) better than TH4 (some experimental results can also be seen in [2], [3] and [5]), but cannot handle KP(1) directly. For testing problems of the SSP type, the number of possible undominated terms after using TH2 is over 14 times than using TH4. The effectiveness of TH2 is sensitive to the relationship between $c_j$ and $a_j$ of the problem (see also [2], [3] and [5]), increasing the correlation relation weakens the effectiveness of the criterion, i.e., increases the number of possible undominated terms.

Using TH4 with KP(1) and KP(2) respectively, the numbers of possible undominated terms of both problems are small. Increasing the number $n$ of variables of KP(1) can increase the number of dominated terms that are eliminated. When $n = 10000$, the average 99.89% of variables in KP(1) can be fixed at zero before using any of the known KP algorithms. Since the modular arithmetic method is used, the effectiveness of TH4 depends on the sizes of $a_1$ and $n$ in a problem: the smaller $a_1$ and larger $n$, especially $a_1 \ll n$, the more dominated terms can be eliminated.

References


