On the Vehicle Routing Problem with Pick-up and Delivery

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Abstract
In this paper we discuss a variant of the well-known Vehicle Routing Problem which involves finding a minimum distance routing of a fleet of vehicles which must service a set of customers. The variation considered is to segregate the customers into two classes: delivery customers which require a load from the vehicle depot, and pick-up customers which have a load to be picked up and taken to the vehicle depot.

Construction and improvement heuristics for obtaining feasible solutions to this problem are presented. A Lagrangian dualization approach for obtaining lower bounds on the optimal solution value is also discussed.

1 Introduction
The Vehicle Routing Problem (VRP) classically involves servicing a set of nodes, representing customers requiring a delivery, using a fleet of homogeneous vehicles domiciled at a Distribution Centre (DC). Each node must be contained in exactly one route and all routes must be feasible with respect to vehicle capacity. The objective is to determine a set of routes where the sum of the distance travelled by the vehicles is minimized.

The variant which we consider is to partition the nodes into two distinct classes:

- Pick-up nodes, which have a load destined for the DC
- Delivery nodes, which require a load from the DC

Note that this is not identical to the Dial-A-Ride problem [1] where the pickup nodes need not necessarily be destined for the DC nor need deliveries emanate from the DC. We make the assumption that there are no access restrictions upon the vehicles so that a pick-up node may be serviced at any time subject only to vehicle capacity constraints (c.f. the Backhauling problem where all delivery nodes must be serviced before any pick-up node). We will refer to this problem as the VRPPD.

The VRPPD requires a modification of the vehicle capacity constraint. In the classical VRP one only need check that the sum of the node demands serviced by each vehicle does not exceed the maximum vehicle capacity. For the VRPPD there must be a more dynamic constraint which maintains vehicle capacity feasibility at each node in every route.

The assumption that Euclidean distances represent a reasonable approximation to actual distances is not always realistic so non-Euclidean distances were simulated by means of an incomplete rectilinear grid. The use of such a grid is an attempt to reduce the number of degrees of freedom associated with a problem instance when comparing methods for solving the VRPPD.
2 Routing

Byrne [2] presented a variety of heuristics to obtain solutions to the VRPPD. The general form of these heuristics was a clustering phase followed by a routing phase. The clustering phase involved choosing a seed node and assigning nodes to that seed whilst the sums of the pick-up loads and delivery loads respectively remained below the vehicle capacity. This forms a cluster of pick-up and delivery nodes to be serviced by a single vehicle. The seed node was selected by a variety of methods and the assignment of nodes was controlled by a load efficiency criterion. This criterion restricted the assignment of a node to a cluster unless the ratio of the load of the node to the distance from the seed to the node is greater than the given load efficiency parameter. This was used to prevent the addition to clusters of a distant node with a small load. The usage of the load efficiency criterion was modified in most of the heuristics due to the generation of excess numbers of clusters (and hence vehicles) when there are few customers remaining unassigned to any cluster. The modifications used were

- Set load efficiency to zero when number of clusters attains minimum required
- Decreasing the load efficiency parameter as the sum of loads assigned to a cluster increases
- A combination of the above modifications

A modified Farthest Insertion heuristic was used to obtain a routing for each cluster created. The modification ensured that vehicle capacity was never exceeded at any point by altering the insertion criteria so that nodes which may violate the vehicle capacity constraint at certain points in the route are not inserted at those points. Such a modification may be made to any insertion-based heuristic. Note that this will always obtain a feasible route from the clusters as the backhauling route is always feasible with respect to vehicle capacity.

Improvement methods developed especially for the VRPPD were applied to the initial solutions provided by the clustering/routing heuristics. These methods included

- Moving a node to another route
- Exchanging two paths between their routes
- Exchanging two paths in a single route

The improvement methods proved to be very efficacious in reducing the total distance travelled by the vehicles even when applied to initial solutions of poor quality in terms of distance. These methods are essentially special cases of the generalised k-opt exchange procedures as described by Potvin et. al. [3] applied to the VRPPD instead of the MTSP. The improvements that we make must be more intuitive than incremental application of the k-opt procedure to the MTSP as they must be cognizant of the tighter capacity constraint of the VRPPD.

An example of the results from the various heuristics on a single problem instance is presented in Figure 1 where the initial and improved solutions are given for 836 combinations of parameters and heuristic methods. This typical example demonstrates the power of the improvement heuristics applied to the initial solutions.
3 Bad-Case Analysis

Worst-case analysis is a means for comparing the performance of heuristics by determining an upper bound on the ratio of a heuristic solution to the optimal solution. It is not always possible however to produce an analysis which holds for all problem instances, a fact demonstrated in the literature by worst-case analyses on heuristics with very restricted problem sets. It is therefore useful to generate a bad-case analysis of a heuristic to provide insight on its performance and give a lower bound on the worst-case ratio.

Figure 2 presents an example problem instance upon which an analysis of the initial routing heuristics may be performed. Here each node (which we assume is a delivery customer as VRPPD is a generalization of VRP) is at a distance $R$ from the DC and is connected to its two nearest nodes cyclicly by arcs of length $d$. We assume that $R > nd$, where $n$ is the number of nodes, each having an identical load of $q$. This is to ensure that the closest node to each node is one adjacent to it on the cycle. We also let $nq = M$ so that all nodes on the cycle may be serviced by one vehicle of maximum capacity $M$.

If we apply a heuristic incorporating the basic load efficiency criterion to this problem, it will attempt to choose a seed node. As the problem is rotationally symmetric, any node may be chosen as a seed. It will then proceed to cluster nodes about this seed node. If the load efficiency parameter, $e$, is too large i.e.

$$e > \frac{q}{d}$$

then no nodes will be added to the cluster so that a single node route will be generated. We can ensure that $e$ is sufficiently large by choosing an appropriate value of $d$. This process
will continue for all nodes, resulting in an initial solution with \( n \) single node routes. However the optimal solution requires only one route. As \( d \to 0 \), the ratio of the heuristic solution to the optimal solution tends to \( n \). As \( n \to \infty \) this ratio tends to \( \infty \), so for this heuristic the bad-case ratio is actually a worst-case ratio.

If the heuristic used is the version which restricts the usage of load efficiency to one less than the theoretical minimum number of vehicles this will considerably reduce the number of single node routes generated. To obtain a bad-case ratio we let \( nq = cM \), where \( c \) is the number of routes required in the optimal solution, and w.l.o.g. let \( n = M(c - 1) + c + 1 \). We then will obtain \( c - 1 \) single node routes and \( c - 1 \) routes each with a full vehicle load. This results in a heuristic solution to optimal solution ratio of

\[
\frac{2R(c - 1) + (c - 1) \left[ \left( \frac{n}{c} - 1 \right) d + 2R \right]}{c \left[ (n - 1)d + 2R \right]}
\]

which tends to the value of 2 when \( d \to 0 \) and \( c \to \infty \). This implies that the worst-case performance of the heuristic will give a solution at least twice the distance of the optimal solution.

If the available capacity modified load efficiency version of the heuristic is applied to the problem in Figure 2, the bad-case ratio will again tend to \( \infty \) as we can always choose appropriate values of \( d \) so that \( e \) is sufficiently large to generate single node routes.

An interesting consequence of this bad-case analysis is that one may examine a problem instance and determine a priori which values of the load efficiency parameter will give solutions containing single node routes. This will occur for node \( j \) when both

1. \( \max_i \left( \frac{\text{load of node } i}{\text{distance from } j \text{ to } i} \right) < e \)
2. \( \max_i \left( \frac{\text{load of node } j}{\text{distance from } j \text{ to } i} \right) < e \).

Note that these criteria will produce a lower bound on the number of single node routes in the initial solution.
4 Lagrangian Relaxation

Another method for determining a bound on the VRPPD is to use Lagrangian Relaxation. A method presented by Fisher [4] relaxes a minimum $k$-tree formulation of the VRP and solves the Lagrangian dual to obtain a lower bound on the optimal solution value. Note that there is no guarantee that the solution obtained via this method will correspond to a valid routing structure. The formulation used was

$$Z^* = \min_{x \in X} \sum_{i=0}^{n} \sum_{j=0}^{n} c_{ij}x_{ij}$$

s.t. \( \sum_{j=1}^{n} x_{ij} = 2 \) \( \forall \ i = 1 \ldots n \) (1)

\( \sum_{i \in S} \sum_{j \in S} x_{ij} \geq 2k(S) \) \( \forall \ S \subseteq N, |S| \geq 2 \) (2)

where $Q$ is vehicle capacity,
$a_i$ is the load of delivery node $i$,
$N$ is the set of nodes (excluding the DC) with cardinality $n$,
$A$ is the set of arcs connecting node set $N$,
$k$ is the number of vehicles,
$X$ = \{x|x = 0 - 1 \}

and defines a $k$-tree on $(N, A)$ with $\sum_{i=1}^{n} x_{0i} = 2k$.

$$k(S) = \left\lfloor \frac{\sum_{i \in S} a_i}{Q} \right\rfloor$$

Constraint set 1 enforces any vehicle that visits a node to leave that node. Constraint set 2 ensures that the number of vehicles entering and leaving any set of nodes is sufficient to service all of the nodes. Recall that a $k$-tree on a graph of $n + 1$ nodes is the $n + k$ arcs that span the graph. The amendments required for VRPPD are

$$k(S) = \max \left( \left\lfloor \frac{\sum_{i \in S} a_i}{Q} \right\rfloor, \left\lfloor \frac{\sum_{i \in S} b_i}{Q} \right\rfloor \right)$$

\( \sum_{i \in P, j \in P} x_{ij} \leq |P| - 1 \) \( \forall \ P \subseteq N, |P| \geq 2, j > i \) (3)

where $a_i$ is the load of delivery node $i$,
$b_i$ is the load of pick-up node $i$,
node set $P$ has an implied route which is infeasible.

The redefinition of $k(S)$ is required due to the introduction of pick-up nodes. Constraint set 3 disallows any routes which are infeasible with respect to vehicle capacity from entering the solution. Note that this is not a complete formulation due to the fact that the sets $P$ are not defined explicitly. This formulation is however more tractable when the two capacity constraints are dualized.
The Lagrangian relaxation of this problem is

\[ Z_D(u, v_S, v_P) = \min_{x \in X} \sum_{i=0}^{n} \sum_{j=0}^{n} c_{ij} x_{ij} + 2 \sum_{i=1}^{n} u_i + 2 \sum_{S \subseteq N} v_S k(S) - \sum_{P \subseteq N} v_P (|P| - 1) \]

where

\[ c_{ij} = c_{ij} - u_i - u_j - \sum_{S \text{ s.t. } i \in S, j \in S} v_S + \sum_{P \text{ s.t. } (i,j) \text{ is part of infeasible route implied by } P} v_P \]

and

\[ u_i \text{ free, } u_0 = 0, v_S, v_P \geq 0, |S| \geq 2, |P| \geq 2. \]

As it is well-known that \( Z_D(u, v_S, v_P) \leq Z^* \), we attempt to maximize \( Z_D \) with respect to \( u, v_S \) and \( v_P \). As there are \( O(2^n) \) constraints associated with the sets of \( S \) and \( P \) we generate these constraints dynamically from any violated sets. Subgradient optimization was then used to maximize the function. The construction of the minimum cost \( k \)-tree was performed by the following steps:

1. Obtain minimum cost spanning tree.
2. Add to it the \( k \) least cost edges not already in tree.
3. If \( k \)-tree does not have degree \( 2k \) on DC then modify it by a sequence of edge exchanges.

Applying this technique to problem instances of VRPPD resulted in a lower bound which was on average around 92% of the best heuristic solution obtained with the best figure being 94.5% of the best known solution. It was observed that the \( v_P \) variables contributed little to the value of \( Z_D \). This was attributed to the different problem structures of VRP and VRPPD. The objective of minimizing distance in the standard VRP also has the effect of minimizing the number of vehicles. Figure 3 gives an example which illustrates this fact. Each of the nodes has a unitary load and the vehicle capacity is four units. The optimal solution when minimizing distance is to use four vehicles, each servicing three pick-up and three delivery nodes. The minimum number of vehicles required however is three, producing a solution with a greater total distance. The number of vehicles is required as input to the Lagrangian

![Figure 3: Problem Structure for Differing Objectives](image-url)
relaxation. If this number is the minimum number required for the VRPPD structure then constraint set 2 is not tightly bound. If the number of vehicles is the fleet size required to service the individual sum of the pick-up or delivery loads (whichever is larger) then constraint set 3 is not tightly bound. This therefore reduces the efficacy of this procedure for obtaining lower bounds to the VRPPD.

5 Conclusions

We have presented a bad-case analysis of the initial routing heuristics for the Vehicle Routing Problem with Pick-up and Delivery. An investigation into the performance of the improvement heuristics when applied to problems of differing grid structure is an area for further research. Preliminary results have shown that our heuristics’ performance is robust with respect to grid arc density. Stochastic aspects (such as demand variation, customer subsets requiring service and vehicle availability) are currently being examined via a simulation study. The Lagrangian relaxation approach has been shown to be limited when applied to VRPPD problem instances where the optimal solution uses more than the minimum required number of vehicles. It remains however a method of determining a lower bound on the optimal solution value.

References


