EXTENSIONS OF THE PETAL METHOD FOR VEHICLE ROUTING

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Abstract: The petal method for vehicle routing has a number of important features which include:

1. For the class of petal routes, the method finds an optimal solution of the routing problem.
2. The optimal petal solution can be found by solving a linear programme which has integral vertices and a naturally integer optimal basic feasible solution.
3. Time windows and fleet composition constraints can be easily incorporated.

Two important disadvantages of the approach are:

1. Many small travelling salesperson problems must be solved.
2. The solution is limited to vehicle routes with strict petal form.

In this paper we show that by extending the definition of a petal route, more general forms of vehicle route can be generated without invalidating the important features of the original petal scheme. It will be shown that the optimal generalised petal solution can be produced very efficiently by multiple application of a shortest path algorithm and that by embedding this process within a tabu search procedure, we can develop an effective method for vehicle routing.

1. Introduction

The Vehicle Routing Problem (VRP) and the related Vehicle Scheduling Problem (VSP) have been the subject of extensive research due to their practical importance in the transportation industry. The VRP involves the construction of vehicle routes to visit a number of delivery points from a central depot. The VSP also includes temporal aspects associated with time windows for visits which impose extra structure on the vehicle routes. The problems are NP-hard so much of the research has centred on the development of heuristics. One type of heuristic involves the construction of routes with a restricted petal-like structure. Petal routes visit all delivery points in a geographic sector centred on the depot. In other words, no customer is left unvisited in the sector. This restriction on the structure of a route reflects the observation that in many problems optimal routes exhibit a petal or near petal structure. Gillet and Miller [1974] developed a heuristic algorithm called the sweep method based on this restricted form of vehicle route. Foster and Ryan [1976] showed that an optimal petal solution can easily be found from the set of all possible petal routes by solving a linear programme in which all basic feasible solutions are naturally...
integer. They also showed that some subsequent relaxations of the strict petal route solution could be considered although the inclusion of non-petal routes destroyed the natural integer properties of the LP making it more difficult to find optimal integer solutions. This paper will discuss a generalization of the Foster and Ryan petal method which permits consideration of non-petal routes without destroying the natural integer properties of the underlying linear programme. We will also show that an optimal generalised petal solution can be found more efficiently by solving a small number of shortest path problems in an underlying directed graph representation of the set of all generalised petals.

2. An Example of the Petal Method

The concept of the petal method will be illustrated using the problem shown in Figure 1. Each of the 13 delivery points has an associated delivery demand shown in brackets. The distances between delivery points are euclidean and all distance calculations are performed using real arithmetic. The goal is to minimise the number of vehicles required to deliver from a central depot, and for that number of vehicles, to then minimise the total distance travelled. Each vehicle has a capacity of 10 units of demand.

![Figure 1 Example Problem](image)

The delivery points have been numbered in radial order about the depot. To generate the set of all petal routes, all radially contiguous subsets are considered as possible routes. The enumeration of the set of all petal routes can be implemented very efficiently as discussed by Foster and Ryan [1976]. A petal route is feasible if the quantity of goods delivered on the route does not exceed the capacity of the vehicle and if the total distance travelled, as determined by the Travelling Salesperson sequence of the deliveries, does not exceed the imposed distance limit. An example of a feasible petal route is shown in Figure 2(a). The petal route in Figure 2(b) is infeasible because the sum of customer deliveries (12) exceeds the capacity of the vehicle (10). The route in Figure 2(c) is not a petal route because the deliveries are not a radially contiguous subset. However, if delivery 1 was added to the subset it would then become a petal route.
Given the deliveries in the radial order 1,2,3,...,13, the enumerated subsets of contiguous deliveries corresponding to all feasible petal routes are:

(1) (1,2) (1,2,3)
(2) (2,3) (2,3,4)
(3) (3,4)
(4) (4,5)
(5) (5,6)
(6) (6,7) (6,7,8)
(7) (7,8) (7,8,9)
(8) (8,9)
(9) (9,10)
(10) (10,11)
(11) (11,12) (11,12,13)
(12) (12,13) (12,13,1)
(13) (13,1) (13,1,2) (13,1,2,3)

Table 1. Contiguous subsets from the natural radial order.

It should be noted that petal generation treats the order as cyclic in that the subsets beginning with deliveries 12 and 13 wrap around to include the deliveries at the beginning of the order. For this reason we will refer to orders as cyclic orders. The cost of a route, considered to be proportional to the distance travelled, is determined by the Travelling Salesperson (TSP) sequence of deliveries (Lawler et al. [1985]). It should also be noted that the delivery sequence in the petal route determined by the TSP will differ in general from the given subset order. For example, the subset (11,12,13) generates a petal route (depot-11-12-13-depot). Each petal route is either a single delivery or a route which results from the addition of the next delivery in the order to the subset of deliveries considered on the previous petal route. Given this sequential building process, it is possible to develop efficient heuristic TSP algorithms which add the extra delivery to the previous route and then attempt to improve the TSP solution.

Given the set of all petal routes, the optimal subset of petal routes covering each delivery exactly once can be easily determined. Foster and Ryan [1970] proposed the use of the LP simplex method for this purpose but in Section 3 we discuss an alternative and more efficient shortest path technique for producing the optimal solution.

The optimal petal solution, shown in Figure 3, requires 5 vehicle routes and a total distance of 68.74. It is clear however that when more general non-petal forms of vehicle route are permitted, an improved solution with total distance of 61.68 can be found as shown in Figure 4.
An important contribution of this paper is to demonstrate that the petal method outlined above can be used to produce the optimal solution shown in Figure 4. All that is required is to reorder the deliveries in a non-radial cyclic order before applying the petal generation scheme. The routes generated will be referred to as *generalised petals*. In geographic terms, the routes will no longer be pure petals but will involve contiguous subsets of deliveries from the non-radial cyclic order. For example, consider the deliveries in the cyclic order 3,7,5,8,10,6,4,2,13,12,11,1,9. Given this order, the set of all subsets of contiguous deliveries corresponding to feasible generalised petal routes are:

<table>
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<th>Subsets</th>
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<td>(9)</td>
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*Table 2. Contiguous subsets for the cyclic order (3,7,5,8,10,6,4,2,13,12,11,1,9)*

It can be seen that the five routes making up the optimal solution of Figure 4 are contained within the set of generalised petals. From an optimization point of view, the determination of an optimal generalised petal solution from the set of all generalised petals is no more difficult than the determination of the optimal petal solution from the set of all petals. The shortest path method discussed in Section 3 can also be applied to the set of generalised petals to find the optimal solution.

It should be noted that there are many different cyclic orders of deliveries which contain the optimal routes of Figure 4 within the corresponding sets of generalised petals. It is sufficient to find a cyclic order in which the deliveries on each optimal route are contiguous. The order of the deliveries within each contiguous subset and the order of the subsets is unimportant. For example, an alternative cyclic order of 2,3,5,7,4,6,8,10,9,11,1,12,13, which is a small perturbation of the natural radial order, will also produce the optimal solution of Figure 4. Provided the optimal generalised petal solution can be found efficiently for a given cyclic order, the vehicle routing problem can be solved by considering many permutations of the radial cyclic order. In Section 4 we discuss the use
of tabu search (Glover [1986]) as a method for controlling the permutation of the cyclic order.

3. Optimal Petal Selection

Given a set of petal routes, Foster and Ryan [1976] select the optimal routes by formulating the problem as a set partitioning problem in which each row of the constraint matrix corresponds to a delivery point and each column to a petal route. Foster and Ryan show that for petal routes, the LP relaxation of the set partitioning problem is totally unimodular and the optimal integer solution can be found by just solving the LP. It can easily be shown that this result remains true for generalised petals. While the LP method has the advantage that extra constraints can be added to model an inhomogeneous fleet or a multi-period problem (Pedder and Philpott [1989]), the LP solution time remains the major bottleneck.

A more efficient method for selecting the optimal petal routes can be developed if the routes are represented by a weighted cyclic digraph \( G = (V, E) \). The optimal petal routes then correspond to a shortest cycle in the digraph. The vertices \( V \) of the digraph correspond to deliveries. The edges \( E \) correspond to generalised petals. Each generalised petal route is represented by an edge from vertex \( i \) to vertex \( j \) where vertex \( i \) is the first delivery from the cyclic order that is in the petal and vertex \( j \) is the first subsequent delivery from the cyclic order that is not on that petal. For example the petal route \((11,12,13)\) in the natural radial order corresponds to the edge originating from vertex 11 and terminating at vertex 1. In Figure 5(a) we show the cyclic petal digraph of all petal routes based on the subsets of Table 1. In Figure 5(b) we show the cyclic petal digraph of all generalised petal routes based on the subsets of Table 2.

![Cyclic Petal Digraphs](image)

**Figure 5**: Cyclic Petal Digraphs

A cycle in the petal digraph is a path that starts at a reference vertex, traverses at least one edge and returns to the reference vertex. An elementary cycle is a cycle that does not bypass the reference vertex. On a cyclic petal digraph, an elementary cycle corresponds to a set of petal routes that will cover all deliveries. For example, the cycle \((1,4,6,9,11,1)\) in Figure 5(a) (described by naming the sequence of vertices visited) corresponds to choosing the petal routes \((1,2,3), (4,5), (6,7,8), (9,10), (11,12,13)\). Thus the problem of finding the optimal petal routes has been transformed into a problem of finding the shortest elementary cycle on the petal digraph. This same approach can also be used for sets of generalised petals corresponding to non-radial delivery orders as shown in Figure 5(b).
The shortest elementary cycle can be found by multiple applications of a shortest path (SP) algorithm applied to a derived acyclic digraph. This derived acyclic digraph is constructed by selecting a reference vertex. The cyclic digraph is then unfolded so that the reference vertex is the source vertex of the resulting acyclic digraph and a new vertex, also corresponding to the reference vertex, is created as a sink. An example of the derived acyclic digraph from Figure 5(a) with vertex 1 as the reference is shown in Figure 6.

Because the new graph is a directed acyclic graph, a very simple and quick SP algorithm can be used to find the SP from the source to the sink. In general, one unfolding and application of the SP algorithm is not sufficient to find the shortest elementary cycle in the original cyclic digraph. When the cyclic digraph was unfolded with vertex 1 as reference, the edges (12,2), (13,2), (13,3) and (13,4) corresponding to petals (12,13,1), (13,1,2) and (13,1,2,3) were not represented in the derived acyclic digraph. Three further unfoldings with vertices 2, 3 and 4 as the reference vertices are required to find the shortest elementary cycle in Figure 5(a). It is possible however to minimize the number of unfoldings required by carefully selecting the reference vertices. For example if we select the vertices 4 and 5 (or 5 and 6, or 6 and 7, or 9 and 10, or 10 and 11, or 11 and 12) just two unfoldings and SP computations would be required. In general, we identify vertex \( j \) for which \( V(j) = \{ k \mid e_{jk} \in E \} \) has minimum cardinality and then choose the vertices of \( V(j) \) as reference vertices.

For a given cyclic order of the deliveries, the multiple SP method for determining the optimum generalised petals is significantly faster than the LP method. However, a disadvantage of the SP method is that the side constraints associated with an inhomogeneous fleet or a multi-period problem become more difficult to accommodate. These difficulties are currently under further investigation.

4. Tabu Search and Order Permutations

In order to generate improved generalised petal solutions, a method for controlling the permutation of the cyclic order is required. The method should provide an escape mechanism from local optimal solutions and also discourage the possibility of "cycling" back to a previously considered order. Tabu search (Glover [1986], Glover [1989], Glover [1990a], Glover [1990b]) is a recently developed meta-heuristic which possesses both these attributes. Since about 1985 it has been successfully applied in a wide variety of applications (see for example Hertz and de Werra [1987] and Lee [1989]). The method of tabu search can be used to guide any process that employs a set of moves for transforming one solution (or solution state) into another; the process should also provide an evaluation function for measuring the attractiveness of these moves. At each step the best move is chosen from a list of candidate moves and the move (or its inverse) is then made "tabu" for a small number of subsequent steps. However, moves which are currently on the tabu list can be chosen as the best candidate move provided the value of the move exceeds an "aspiration level" which is set to recognise especially valuable moves despite their current tabu status. In general the tabu restrictions and aspiration criteria have the goal of
preventing the reversal of recently made moves and in doing so they enable the method to move away from local solutions.

There is considerable flexibility in the design and implementation of tabu search methods. Even in the simplest applications the definition of a move, the choice of a move evaluator, the nature of the tabu characteristics, the administration of the tabu list and the choice of aspiration criteria all require careful consideration. In the context of the vehicle routing application and the permutation of cyclic orders outlined in Section 3, we are currently investigating these aspects of the tabu search method. The importance of the underlying geographic radial ordering will obviously have considerable influence on the design of our tabu search method. For example a tabu search move might be to move one or more contiguous deliveries from their present position to a new position in the cyclic order. The number of deliveries moved and the distance of the move in the order both need further specification. Moves could be restricted to “preserve” aspects of the underlying radial order. For example, certain deliveries at large distances from the depot could be treated as fixed points in the order since they could be thought of as imposing a sequencing of the optimal routes within the order. It is also clear that deliveries close to the depot will be more likely to move further from the radial order. In defining candidate moves, it is also reasonable to treat as a group subsets of deliveries which are likely to appear together on a vehicle route. The impact of candidate moves on the resulting generalised petal digraph could also influence their definition. For example the repositioning of a single delivery from its present position in the cyclic order to a new position within an adjacent petal will generate a rather local effect on the set of generalised petals. A small number of edges in the cyclic digraph will be deleted and replaced by those corresponding to the new generalised petals. The deleted edges and the new edges will all correspond to generalised petals involving the single moved delivery. It is clear then that if candidate moves involve small permutations of the cyclic order it will be reasonably simple to update the cyclic digraph and recompute the shortest paths to determine the new optimal generalised petals. More extensive permutations of the order however will result in more extensive modifications of the cyclic digraph and will obviously involve much more computation to determine TSP solutions for a large number of new generalised petals.

5. Summary

In this paper we have shown that:
1) the optimal solution of a VRP corresponds to a family of "optimal cyclic orders",
2) for a given cyclic order the optimal generalised petal set can be determined very efficiently and
3) tabu search provides a heuristic framework within which permutations of the cyclic order can be considered.

Results from an initial version of the method based on these properties are encouraging and indicate that it should efficiently produce good quality solutions to the VRP. Although we have used the VRP to illustrate the concepts of this approach the ideas can also be extended to include more complex problems such as the VSP and the Multiple Depot Vehicle Routing Problem.

6. References


Glover, F., "Tabu search Part II", ORSA Journal on Computing 2(1990a)