AN APPROXIMATE SOLUTION TO THE (s,Q) INVENTORY CONTROL PROBLEM FOR GAMMA DISTRIBUTED LEAD TIME DEMAND

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ABSTRACT

Using a simple approximation for the safety stock when the lead time demand has a gamma distribution, a closed form solution to the (s, Q) inventory control problem is obtained. In most cases, use of the above-mentioned approximation gives better results than using normal and Laplace distributions as approximations to gamma distributions for the purpose of modeling lead time demands. The main limitations of the technique result from the limitations of the model on which the technique is based. Methods of dealing with this problem are given.

1. THE BASIC TECHNIQUE

In an (s, Q) inventory control system, a quantity Q is ordered when the inventory position (stock on hand plus dues in minus dues out) falls below the reorder point s.

The total of the annual holding and ordering costs is often approximated using

\[ T(s, Q) = \frac{Q}{2} V C_h + \frac{Q}{Q} C_r \]  

(1)

with \( s = \mu + SS \)  

(2)

where SS = safety stock, 
V = value of the item ($/unit), 
\( \mu \) = mean annual demand, 
\( C_h \) = holding cost ($/$/year), 
\( C_r \) = ordering cost ($/order) and 
\( \mu \) = expected lead time demand.
But \( SS = k \sigma \) \( (3) \)

where \( k \) = safety factor required to meet the required service level

and

\( \sigma \) = standard deviation of lead time demand.

The approximation

\[
\frac{Q}{\sigma} (1 - P_2) = \int_{k+\mu/\sigma}^{\infty} (x - k) f(x) \, dx
\]

\( (4) \)

is widely used where \( P_2 \) is the fraction of demand supplied immediately from stock and \( f(x) \) is the distribution of the ratio of the lead time demand to \( \sigma \).

If the lead time demand has a gamma distribution then, subject to the appropriateness of equations (1) and (4), \( k \) can be approximated by

\[
k = \alpha - 6, \ln \left( \frac{Q}{\sigma} \cdot \frac{1 - P_2}{0.05} \right)
\]

\( (5) \)

where

\[
\alpha = a \phi + b
\]

and

\[
\phi = \frac{c}{\mu} + d
\]

where

\[
a = 0.371,
b = 0.275,
c = 0.496
\]

and

\[
d = 0.517
\]

Because of the perceived difficulties of using the gamma distribution as a model of lead time demand, Silver and Peterson [2] suggested the use of a normal or Laplace distribution, depending on the value of \( \mu \). In most cases, equation (5) provides a more accurate estimate of \( k \) than would be obtained if a normal or Laplace distribution were to be used as an approximation to the gamma distribution.

Using equations (1), (3) and (5), the optimal value of \( Q \) is given by

\[
Q = \beta + \sqrt{\beta^2 + CLOQ^2}
\]

\( (6) \)

where

\[
\beta = \alpha \delta
\]

\( (7) \)

and CLOQ is the classical economic order quantity. The corresponding value of \( s \) can be obtained using equations (2), (3) and (5).
The technique described above is an extension of one which Mabin [1] developed for normally distributed lead time demand.

In circumstances in which the equations (1) and (4) are appropriate, the above-mentioned approximate solution is usually very close to optimal as far as the total cost is concerned. However, for a large proportion of real life inventory control problems, the results are very poor. There are two reasons for this, viz.
(a) The assumptions of the model are frequently violated.
(b) The approximations involved in equations (1) and (4) are frequently inappropriate. In equation (1) it is assumed that $Q$ or $SS$ is large compared with $\sigma$ and in equation (4) it is assumed that $Q-\mu$ is large compared with $\sigma$. As a result, "optimal" solutions obtained using equations (1) and (4) tend to be far from optimal in cases in which one or, especially, more than one of the following conditions applies:
(i) The demand rate is low.
(ii) The annual usage value is high.
(iii) The mean lead time is long.

2. MORE GENERALLY APPLICABLE TECHNIQUES

After improving the approximations involved in equations (1) and (4) it was found that much better results could be obtained by replacing equation (7) by

$$\beta = \alpha s P_2^3 \left[ 1 - \exp \left( - \frac{C\theta Q^2}{4\mu^2} \right) \right]$$

and obtaining $s$ from

$$s = \sigma \sum_{i=1}^{7} \sum_{j=1}^{4} a_{ij} \left[ \frac{\max(Q_i,\mu)}{\sigma} (1 - t_j^2) \right] e^{-j^2}$$

where each $a_{ij}$ is a constant. Further improvement to the approximation involved in equation (4) will make it possible to improve upon equations (8) and (9).

In situations in which the assumptions involved in equations (1) and (4) are seriously violated and there are no sub-optimal relative minima of the total cost function, one method of obtaining a near optimal solution is as follows:
(i) Apply equations (8) and (9) and call the solution $(s_1, Q_1)$.
(ii) Choose another pair $(s_2, Q_2)$ such that $s_2 > s_1$ and $Q_2 > Q_1$.
(iii) Change $C_{P}$ and $C_{F}$ so that average of the values of $\frac{\partial T}{\partial Q}$ at $(s_1, Q_1)$ and $(s_2, Q_2)$ is the same as it would be if the assumptions involved in equations (1) and (4) were valid and similarly for $\frac{\partial T}{\partial s}$.
(iv) Apply equations (8) and (9) and call the solution $(s_1, Q_1)$.
(v) If $s_1$ and $Q_1$ changed little in Step (iv) then stop; otherwise go to Step (ii).

REFERENCES
