MULTIPLE OBJECTIVE MATHEMATICAL PROGRAMMING: A REVIEW

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SUMMARY

Since the early 1970's, a number of techniques have been developed for solving decision problems with multiple objectives. This paper reviews the major solution methods in the area of multiple objective mathematical programming. Greater emphasis is placed on the basic concepts underlying these methods than on mathematical detail.

1. INTRODUCTION

Man will always be faced with the need to make decisions, decisions which must be made in the midst of a complex environment. And most of these decisions will have multi-dimensional consequences; i.e., a number of different criteria will be simultaneously affected by the decision. The multi-dimensional nature of decision problems tended to be avoided in the early days of management science, as most effort was directed toward single objective models. However, since the early 1970's, a considerable upsurge in research has occurred in a number of different directions all of which consider a multiplicity of criteria or objectives. This paper, as indicated by the title, will focus on one such research direction known as multiple objective mathematical programming (MOMP). Other reviews of the literature include Hwang and Masud [1979], Chankong and Haimes [1983] and Evans [1984].

It is the intention of this paper to focus on the concepts of (MOMP) and especially of the solution
methodologies, rather than on precise mathematical detail. After introducing necessary and relevant terminology, the remainder of the paper will deal will the developments in solution methodologies.

2. DEFINITIONS AND TERMINOLOGY - THE MOMP

The MOMP can be stated mathematically in terms of continuous decision variables. The general form of the model is

\[
\text{"MAX" } F(x) = \{f_1(x), f_2(x), \ldots, f_q(x)\}
\]

subject to \( x \in X \)

where \( x \) is an \( n \)-dimensional Euclidean vector, \( x = (x_1, x_2, \ldots, x_n) \), \( X \) is the set of feasible decisions, \( X = \{x: x \in \mathbb{R}^n, g_i(x) \leq 0, i = 1, 2, \ldots, m\} \), \( F(x) \) is a vector of scalar valued objective functions defined on \( x \), and "MAX" is defined in Section 2.1.

The set of feasible solutions to (1) is described by \( m \) continuous functions of the decision variables. This is in contrast to multiple attribute decision models (as they are often referred to in the literature), where the set of feasible solutions consists of a countably small number of discrete alternatives. Rietveld's [1980] study of eight development alternatives for the Maasvlakte area in the Netherlands provides an example where a few, well defined alternatives constitute the entire feasible set. Approaches for solving this problem differ considerably from those developed to solve the MOMP, and will not be discussed in this paper.

2.1 A Definition of "MAX"

The existence of a unique optimal solution to (1) is unlikely, except in the trivial case where a solution \( x^O \in X \) maximizes each and every objective \( f_k(x), k = 1, 2, \ldots, q \). Since such a solution cannot usually be found, the term "MAX" does not retain its traditional meaning (Rosenthal [1982]). For any \( (x^1, x^2) \in X \), the relationship between \( F(x^1) \) and \( F(x^2) \) is not simply "greater than", "less than" or "equal to", since comparisons are required to be made across different and often incommensurable objectives. The solution to the decision problem as stated in (1) is therefore a set of solutions, which are called efficient or
non-dominated solutions.

2.2 Dominance and Efficient Solutions

A definition of dominance is given below.

\[ (x^1, x^2) \in X \text{ } x^1 \text{ dominates } x^2 \text{ if } \]

\[
\begin{align*}
& f_j(x^2) < f_j(x^1) \text{ for some } j \in \{1, 2, \ldots, q\} \\
& f_k(x^2) \leq f_k(x^1) \text{ for all } k \neq j
\end{align*}
\]

The efficient or non-dominated set consists of all feasible solutions to (1) which are not dominated by any other feasible solutions in \( X \). Let \( N \in X \) be the set of efficient solutions. Then for any \( x^0 \in N \), it is not possible to move to another \( x^P \in N \) without decreasing at least one objective function value. Geoffrion (1968) has extended this definition of (2) to a "properly efficient solution", which requires that the marginal gain for any one objective must be bounded relative to marginal losses in the other objectives.

Considerable research effort has been directed toward solution methods which ensure that only efficient solutions to (1) are generated. In fact, in many MOMP solution methods the actual optimization involves nothing more than distinguishing between efficient and inefficient solutions.

A useful characterization of an efficient solution has been given by Soland (1979). Let \( h \) be any function defined on \( \mathbb{R}^q \) which is strictly increasing on any of its components. For \( b \in \mathbb{R}^q \) define

\[
P(h, b) = \max_{x \in X} h(F(x))
\]

subject to \( F(x) \geq b \).

If \( x^* \) is an optimal solution to \( P(h, b) \), then \( x^* \) is efficient. This characterization effectively encapsulates two of the major approaches for solving MOMP's. The first is to optimize a composite objective function (usually an additive form) subject to the constraint set, while the second optimizes a single objective subject to constraints on the achievement of all other objectives. These two forms are detailed below.
\[
\begin{align*}
\text{Max} & \quad \sum_{k=1}^{q} w_k f_k(x) \\
\text{subject to} & \quad x \in X \\
& \quad w_k \geq 0, \quad k = 1, 2, \ldots, q
\end{align*}
\]

and

\[
\begin{align*}
\text{Max} & \quad f_j(x) \\
\text{subject to} & \quad f_k(x) \geq b_k, \quad k = 1, 2, \ldots, q, \quad k \neq j \\
& \quad x \in X
\end{align*}
\]

Using Soland's characterization, a single efficient solution can be generated by assigning values to the parameters \(w_k\) and \(b_k\).

Alternatively, the MOMP can be solved to find all efficient solutions; this is known as the vector maximum approach. However, since the efficient set contains an infinity of solutions, some clarification is necessary. The vector maximum approach has been developed for the situation where all constraints and objectives to (1) are linear, and it finds all efficient extreme point solutions (which are finite in number). It is then possible to describe the infinity of non-extreme efficient solutions in terms of linear combinations of these extreme point solutions.

### 2.3 Matrix of Extreme Solutions and the Ideal Point

The matrix of extreme solutions is given by

\[
P =
\begin{bmatrix}
 f_1(x_1^*) & f_2(x_1^*) \\
 f_1(x_2^*) & f_2(x_2^*) \\
 & \quad \vdots \\
 f_1(x_q^*) & f_2(x_q^*)
\end{bmatrix}
\]

where \(x_k^*\) is the optimal solution to

\[
\begin{align*}
\text{Max} & \quad f_k(x) \\
\text{subject to} & \quad x \in X
\end{align*}
\]

Some authors define the ideal solution \(\bar{y} =\)
\((U_1, U_2, \ldots, U_q) = (f_1(x_1^{\ast}), f_2(x_2^{\ast}), \ldots, f_q(x_q^{\ast}))\). This corresponds to the diagonal of \(P\) which represents the maximum possible achievement of each objective \(f_k(x)\).

The ideal solution is often used as a point of reference in MOMP solution methods, where a distance measure can be used to indicate how "far" the current solution is from the ideal solution.

2.4 Decision Space and Objective Space

In MOMP a distinction is often made between decision space and objective space. Each solution \(x \in X\) can be represented in terms of the decision variables \((x = (x_1, x_2, \ldots, x_q))\) or in terms of the objective function values of those variables \((F(x) = (f_1(x), f_2(x), \ldots, f_q(x)))\). The solution methods to be reviewed all "operate" in objective space since it will usually be of considerably lesser dimension than the decision space.

2.5 Tradeoffs

Tradeoffs are a commonly used concept in MOMP. In essence they are the relative changes in objectives when moving from one feasible solution to another, i.e., they are a measure of the difference between two solutions in objective space. Haimes and Chankong [1979] make the following useful distinction. Consider two feasible solutions \((x^o, x^*) \in X\). Define \(T_{kj}(x^o, x^*)\) as the ratio of change in \(f_k\) to change in \(f_j\). Thus

\[
T_{kj}(x^o, x^*) = \frac{f_k(x^o) - f_k(x^*)}{f_j(x^o) - f_j(x^*)}.
\]

Then \(T_{kj}\) is a pairwise tradeoff if all \(f_p(x^o) = f_p(x^*)\), \(p = 1, 2, \ldots, n\), \(p \neq j, k\), and \(T_{kj}\) is a total tradeoff if there exists at least one \(p\) such that \(f_p(x^o) \not< f_p(x^*)\). (The symbols "\(\not<\)" have the meaning "not equal to".)

Tradeoffs can also be expressed in terms of a direction of movement. Let \(d^* = x^o - x^*\) be the direction in moving from \(x^*\) to \(x^o\), and \(\alpha\) be the distance moved in that direction, \(x^o = x^* + \alpha d^*\). Then the total tradeoff rate at \(x^*\) along \(d^*\) can be defined as
Some methods for solving (1) seek to choose a direction $d^*$ such that only partial or pairwise trade-offs are used, while others utilize the concept of the total tradeoff. This concept of sacrificing an amount of one objective to achieve more of another is central to MOMP, especially with respect to interactive solution methods.

3. DEFINITIONS AND TERMINOLOGY - THE DM's PREFERENCES

Up to this point, no mention has been made of the role of the decision maker (DM) in finding a solution to the MOMP. In the absence of any participation from a DM, the actual solution to the MOMP is not a single solution, but rather a set of solutions. A value judgement (i.e., subjective information) is required from the DM before a single "best" solution can be found. For the case of a single DM, the MOMP can therefore be restated as

"find an $x^* \in X$ such that the most preferred values of $F(x^*)$ are obtained."

3.1 Utility or Value Functions

According to classical economics, the preferences of a rational DM are those of a utility maximizer, i.e., a DM is able to search among the set of feasible solutions and choose that solution which provides the greatest satisfaction or utility. Utility theory therefore assumes that, for an individual DM, there exists a scalar measure of preference for each $x \in X$ which is the DM's utility function.

There are certain conditions which the DM's preferences must satisfy for a utility function to be defined on them. The DM must be able to express both consistent preferences and consistent beliefs, and these beliefs (what the DM thinks is going to happen) are to be independent of preferences (what the DM would like to happen) (Hogarth [1980, Chapter 4]). Consistent preferences imply transitivity, i.e., if $A$ is preferred to $B$ and $B$ to $C$, then $A$ is preferred to $C$. Consistent
beliefs imply that predictive judgements regarding the occurrence of events can be formulated as probabilities, which means that there exist lotteries for which certainty equivalents can be derived, (see, Keeney and Raiffa (1976, pp142-1483)).

Let $V[F(x)]$ be a utility function which represents the preferences of a DM (as scalar values) over the set of feasible solutions. The MOMP can be reformulated to find the most preferred solution by solving

$$\text{Max } V[f_1(x), f_2(x), \ldots, f_q(x)]$$

$$\text{subject to } x \in X (10)$$

In (10) all objectives have been aggregated into a single scalar measure which is then optimized to find the solution $x^*$ of maximum utility. This approach reduces the original multiple objective problem to a new single objective problem. The practical difficulty with this approach is the determination of a suitable form for $V$.

### 3.2 Marginal Rate of Substitution

In contrast to the tradeoffs mentioned in Section 2.5, the marginal rate of substitution (MRS) is the value of a tradeoff according to a DM's utility function, rather than according to the geometry of the feasible set $X$. The MRS is a pairwise tradeoff, not a total tradeoff. With respect to a utility function $V$, $MRS_{kj}$ is defined as the amount of $f_k$ that a DM is willing to sacrifice to acquire an additional unit of $f_j$, at any given point $f^0$ in objective space, i.e.,

$$MRS_{kj}(f^0) = \frac{\delta V[f^0]}{\delta f_j} / \frac{\delta V[f^0]}{\delta f_k}$$

$$= - \frac{df_k}{df_j} \text{ for fixed utility} (11)$$

Figure 1 shows both tradeoff and MRS values. This figure shows the set of feasible solutions in objective space (only two objectives) with the DM's utility function superimposed on top for certain fixed levels of utility. The efficient set is defined by the line segments BC and CE and the ideal point is $(f_1^*, f_2^*) = (10, 6)$. $D$ is the feasible (and efficient) solution of maximum utility where MRS is equal to the pairwise tradeoff.
3.3 Monotonicity of Preferences

A further assumption which is usually made in the context of the MOMP is that the DM's preferences are a monotone function of the level of each objective function. A DM is assumed to always prefer more to less; his or her satisfaction does not decrease as \( f_k \), \( k \in \{1, 2, \ldots, q\} \) increases. Therefore, the marginal value of an extra unit of objective \( f_k \), \( k \in \{1, 2, \ldots, q\} \) is always greater than or equal to zero.

The consequence of this monotonicity of preference assumption is that the DM's most preferred solution will always be efficient. This accounts for the large emphasis placed on the generation of efficient solutions to the MOMP (1). In general, the assumption is a reasonable one if the objectives are appropriately defined.

4. METHODS OF SOLUTION

MOMP solution methods vary according to the characteristics of the problem formulation (e.g., linear or non-linear and size) and according to the information provided by the DM. As was mentioned in Section 3.3, all solution methods are effectively seeking a match
between the respective geometries of the DM's preferences and the efficient set. Some methods require that the DM provide information about the geometry of his or her preferences which is then analyzed by the method. Alternatively, the method provides information (e.g., tradeoffs) which the DM then analyzes. Most solution methods fall somewhere in between.

Several approaches have appeared for classifying solution methods. This paper will follow Hwang and Masud [1979] who have classified solution methods into three categories according to the timing of information provision. These are

- a priori - before solution
- progressively - during solution (interactive methods)
- a posteriori - after solution.

5. METHODS WITH A PRIORI ARTICULATION OF PREFERENCES

Relatively few solution methods fall into this category where subjective information is first elicited from the DM and is then utilized to find a preferred solution.

5.1 Goal Programming

Goal programming (GP) was the first truly multiple objective solution method developed (Charnes and Cooper [1961]), and is based around the intuitive concept of goal setting. Specifically, the DM assigns a goal or target to each objective and then seeks to minimize the deviations from each goal. These deviations, which represent both over and under achievement of goals, are then weighted by the DM so as to reflect their relative importance. GP is a well known technique and will not be considered here in further detail; see Rosenthal [1983] for a critique of GP.

The GP approach has been extended to methods which involve interaction with the DM, e.g., Dyer [1972] and Masud and Hwang [1981].

5.2 Surrogate Worth Tradeoff (SWT) Method

This method is based on the previously mentioned characterization of an efficient solution (5), which is known as the e-constraint formulation, i.e.,
Max $f_j(x)$

(12) s.t. $f_k(x) \geq e_k$, $k = 1, 2, \ldots, q$, $k \neq j$

$x \in X$.

Developed by Haimes and Hall (1974), the SWT method utilizes the concept of a pairwise tradeoff between two objectives. It can be shown that for $F$ differentiable and $\pi_{jk}$ being the Lagrange multiplier for constrained objective $k$, the pairwise tradeoff $t_{jk}$ is defined as

(13) $t_{jk} = \frac{df_j}{df_k} = -\pi_{jk}$

In this method the DM is required to assign a value to various pairwise tradeoffs which are presented to him or her. Specifically, for different efficient solutions the DM assigns a value $w_{ij}$ to each pairwise tradeoff $\pi_{ij}$ for $i, j \in \{1, 2, \ldots, q\}$. $w_{ij}$ are called the surrogate worth functions and are ordinal in nature. If $w_{ij} > 0$ then it is assumed that $MRS_{ij} > \pi_{ij}$. Hence a tradeoff is favoured, i.e., it pays to decrease $f_i$ and increase $f_j$. For $w_{ij} < 0$, the reverse applies. After the DM has provided this information for a number of efficient solutions, a solution in objective space is determined whereby $w_{ij} = 0$ for $i = 1, 2, \ldots, q$, $i \neq j$. This solution is at a point of indifference.

The solution values at this point of indifference are used to constrain the $q-1$ objectives in (12). Then (12) is solved by optimizing with respect to the remaining objective $f_j$. The resulting solution should be the most preferred solution, where the DM has provided these worth assessments prior to solving (12).

Often it is not specified exactly how the indifference solution, where all $w_{ij}$'s are zero, is calculated. This, coupled with the difficulty that the DM often has in actually providing worth values, are the main drawbacks of the method. There is also a more subtle problem. At a solution which is not properly efficient, some Lagrange multipliers will be zero, and pairwise tradeoffs will therefore not be defined. Consequently, if a large portion of the efficient surface is not properly efficient in a MOMP, all improperly efficient solutions will not be included in the pairwise tradeoffs presented to the DM. The resulting solution is less likely, therefore, to be globally preferred.
As GP methods have been made interactive, so also has the SWT method (see Chankong and Haimes [1978], Goicoechea et al. [1982, pp143-149]). GP and the SWT method, along with utility function assessment, are the main solution methods where information is provided by the DM prior to solution. The extension of these two methods to an interactive form is likely to be indicative of the unsuitability of a priori information provision.

6. METHODS WITH PROGRESSIVE ARTICULATION OF PREFERENCES

The majority of MOMP solution methods belong in this second category. The intent of these methods is that via interaction and progressive revelation of preferences, a sequence of solutions will result. This sequence of solutions should, after a finite number of iterations, converge to the most preferred solution of the DM.

6.1 STEM Method

The STEM method, proposed by Benayoun and his colleagues [1971] for linear MOMP's, was one of the first interactive solution methods to be developed. It is conceptually simple, with the preferences of the DM being implicitly incorporated into the solution method by the setting of bounds on the objectives. The MOMP is formulated using a weighted Tchebycheff formulation (see Section 6.8) where the weights are a measure of the relative discrepancy between the maximum and minimum values of \( f_j(x) \). A solution \( x^O \) (in objective space) is presented to the DM who is then required to specify an amount by which a satisfactory objective \( f_j \) is to be relaxed in order to allow improvement in the other objectives. Therefore, at each iteration, the constraint set is augmented by a constraint of the form

\[
f_j(x) \geq f_j(x^O) - \delta_j
\]

where \( \delta_j \) is the amount of relaxation. The weights corresponding to \( f_j \) are then set to zero and the next iteration begins. Since the DM sets bounds for a different satisfactory objective at each iteration, the procedure should terminate after \( q-1 \) iterations with either the most preferred solution or the message that there is no solution acceptable to the DM.

A number of extensions to this approach have been
proposed. These include Belenson and Kapur [1973] who use a two-person zero-sum game approach to determine the appropriate weights at each iteration, and a goal programming extension by Fichefet [1974].

6.2 Method of Geoffrion, Dyer and Feinberg [1972] (GDF)

In contrast to the STEM method, the GDF method places a greater burden on the DM with regard to the provision of information. The DM is required at each iteration to provide a MRS value for different objective pairs. The method assumes that, at least implicitly, the DM possesses a utility function defined on the q objective functions. The MOMP simply becomes

\[
\text{Max } \mathcal{V}(F(x))
\]

subject to \( x \in X \).

It is solved by utilizing the linear approximations of the Frank-Wolfe algorithm. There are two steps at each iteration; finding a best direction of improvement and finding a best step size in that direction. At a feasible (although not necessarily efficient) solution \( x_1 \), the direction finding problem for (15) reduces to

\[
\text{Max } \sum_{k=1}^{q} w_k(x^i) \text{grad}(f_k(x^i)) \cdot y
\]

subject to \( y \in X \)

where \( w_k(x^i) = \text{MRS between } f_k \text{ and an arbitrary reference objective } f_j \). The neat thing about this method is that the exact form of the DM's utility function is not required. Provided that the DM is able to give MRS information at each iteration (consistent with his or her utility function), the direction of best improvement \( d^1 \) can be found. In order to find the step size \( \alpha \), various values of

\[
F(x^i + \alpha d^i) \text{ for } 0 \leq \alpha \leq 1
\]

are presented to the DM, who chooses the most preferred one. The next feasible solution \( x^{i+1} = x^i + \alpha d^i \) is found and the iterations continue.

The sequence of solutions which can begin at any
feasible solution, will not necessarily be efficient until the final, most preferred solution is reached. At each iteration, the onus is on the DM to provide MRS information which is then analyzed by the method. As regards possible disadvantages of the method, Hwang and Masud [1979, p121] comment that the DM often has difficulty in providing MRS information at each iteration and where there are more than two objectives, the choice of a suitable reference objective can be difficult. A considerable advantage of the method is that some inconsistent responses from the DM will not prevent the most preferred solution from being found; they only slow the progress toward it.

A number of other solution methods which are similar in spirit to the GDF approach and yet exhibit further extensions in concept will also be briefly reviewed.

6.3 Efficiency Projections - Winkels and Meika [1984]

This approach generates only efficient points at each iteration. Once the direction of best improvement has been found from the MRS information of the DM, it is projected onto the efficient surface, where these projections are found using a parametric programming approach. This information is presented graphically to the DM who then chooses the appropriate step size. The method is exactly that of the GDF method, except that the initial and all subsequent solutions are efficient.

6.4 Proxy Approach - Oppenheimer [1978]

Where the GDF method uses linear approximations to the DM's true utility function at each iteration, Oppenheimer makes use of an assumed proxy utility function to do the approximating. The MRS information provided at each iteration by the DM is sufficient to evaluate the parameters of an assumed proxy utility function. Given this, it is reasoned that a proxy utility function should be a better local approximation to the DM's true utility function than a linear approximation as used by GDF. The proxy utility functions are not intended to be globally valid; in fact a different proxy will be evaluated at each iteration. Two standard form proxy utility functions used by Oppenheimer are
sum-of-exponentials: $V(F) = \sum_{k=1}^{Q} a_k \exp(-w_k f_k)$

(18)

sum-of-powers: $V(F) = -\sum_{k=1}^{Q} a_k (M_k+f_k)^{\alpha_k}$, $M_k+f_k > 0$

The choice as to which form of proxy function is actually used is at the discretion of the analyst. This proxy approach has a considerable advantage over the GDF method since both the best direction of improvement and the step size are calculated from the local proxy utility function at each iteration.

6.5 SPOT Method - Sakawa (1982)

It is necessary to briefly introduce the interactive surrogate worth tradeoff (ISWT) method (Chankong and Haimes [1978]) in order to illustrate the SPOT method. The ISWT method is an interactive scheme based on the e-constraint formulation (12). It uses Zoutendijk's method of steepest descent to determine a direction of improvement (which is found from the worth values provided by the DM), and a step size at each iteration. At a given solution $x^1$, surrogate worth values $w_{jk}$ are assigned by the DM to each tradeoff $\pi_{jk}$. The updated right hand side of each constrained objective $k$ is simply given by

(19) $e_{k}^{l+1} = e_{k}^{l} + \alpha^{l}(w_{jk}: f_{k}(x^{l}))$

where $\alpha$ is the step size. The similarity with the GDF method can be easily seen; the $w_{jk}$ values contain the necessary MRS information. This procedure will converge to the most preferred solution under the assumption of an ideal DM (who conforms to his or her implicit utility function).

The SPOT method uses the same basic approach as the ISWT method except that (like Oppenheimer) a proxy utility function is used to determine the best direction of improvement. Specifically, if $m_{jk}^{l}$ is the MRS_{jk} based on the DM's proxy utility function at solution $x^{l}$, then the updating equation is

(20) $e_{k}^{l+1} = e_{k}^{l} + \alpha^{l}(m_{jk}^{l}-\pi_{jk})$

Both SPOT and Oppenheimer's proxy approach require consistency checks to be made of the DM's preferences with regard to the particular proxy function which is
being assessed. If discrepancies exist beyond a certain prespecified tolerance level, the inconsistency is explained to the DM so that the tradeoffs can be reassessed and the discrepancy resolved.

Sakawa and Seo [1983] have further extended the SPOT method by using fuzzy set theory. They assume that the DM has a local, but imprecise knowledge of his or her utility function. Instead of the DM having to specify an exact value for each assessed MRS, he or she is required to provide four values. These four values are the absolute minimum and maximum for the MRS and the minimum and maximum of a totally acceptable interval for the MRS. This information is processed using the theory of flat fuzzy numbers. Here an attempt is made to accommodate some of the imprecision expected from a DM, and therefore there must also be some tolerance of inconsistency.

6.6 Tchebycheff Norm Approach - Sakawa and Mori [1983]

Instead of using the e-constraint formulation as in the SPOT method, in this method Sakawa and Mori use a weighted Tchebycheff norm formulation (see Section 6.8). A direction of improvement \( \mathbf{d} \) is found using MRS information provided by the DM. With a step size parameter of \( \alpha \), solutions in this direction \( (f(x^*) + \alpha \mathbf{d}) \) are then found, except that these solutions are first transformed into efficient solutions by using the Tchebycheff formulation. This approach is similar in principle to that of Winkels and Meika [1984] where all solutions in the direction of improvement are projected onto the efficient surface. The difference lies with the actual projection mechanism.

6.7 Method of Zionts and Wallenius [1976,1982] (ZW)

In order for this method to be of practical use for solving the MOMP of (1), it is required that all functions be linear. Furthermore it is assumed that the DM’s underlying or implicit utility function is a linear combination of the objectives. Thus the problem to be solved is

\[
\begin{align*}
\text{Max} & \quad V = \sum_{k=1}^{q} w_k f_k(x) \\
\text{subject to} & \quad x \in X \\
& \quad \sum_{k=1}^{q} w_k = 1, \quad w_k \geq 0, \quad k = 1,2,\ldots,q.
\end{align*}
\]
At each iteration a set of weights \( \mathbf{w} = (w_1, w_2, \ldots, w_q) \) is determined until the process terminates with the weighting structure which maximizes the linear utility function. The most preferred solution will be an extreme point of the efficient set because the composite objective in (21) is linear.

At each iteration, the DM is required to assess a number of total tradeoffs (as distinct from pairwise tradeoffs) and provide an ordinal response of "yes", "no" or "indifferent". When compared with the GDF approach, the burden placed on the DM as regards providing information is affected in two ways. It is lessened in that only ordinal rather than cardinal information is required, but is increased in the sense that the DM has to assess the tradeoff holistically, i.e., over all objectives simultaneously. A weighting structure is then found which is consistent with the DM's responses. (For further details of the method, see de Samblanckx et al. [1982].)

Effectively, the ZW method cuts away a portion of the objective space at each iteration. Consequently, many solutions can be implicitly eliminated at each iteration. The disadvantage is that wrong or inconsistent answers may eliminate preferred solutions.

More recently, Stewart [1984] has proposed a modification to the ZW method whereby provision is made for inconsistent choice behaviour. The problem of finding the vector of weights \( \mathbf{w} \) at each iteration is approached by maximizing the following log likelihood function.

\[
L(\mathbf{w}; \mathbf{S}_n) = - \sum_{\mathbf{S}_n} \log \left( 1 + \exp \left( -\sum_{k=1}^{q} w_k v_{kj} \right) \right)
\]

where \( \mathbf{S}_n \) is a set of pairwise preference statements given by the DM. While there are some additional features to Stewart's logistic regression approach, the basic concept is to allow for inconsistencies by using maximum likelihood estimation. However despite Stewart's own comment that "the Zionts-Wallenius method is quite often relatively insensitive to response errors" (p.1077), his extension to the method represents another attempt to reduce the requirements placed on the DM as information is progressively elicited from him or her.
6.8 The Weighted Tchebycheff Approach (Steuer and Choo [1983])

This approach follows logically from much of Steuer's earlier work (Steuer [1976, 1977]). In this earlier work, Steuer showed how the size of the efficient set could be reduced if the DM is able to a priori specify bounds on the weights as in the following weighted sum formulation.

\[ \max \sum_{k=1}^{q} w_k f_k(x) \]

subject to \( x \in X \)

\[ \sum_{k=1}^{q} w_k = 1 \]

\[ w_k \in (m_k, u_k) , \quad 0 \leq m_k \leq u_k \leq 1 \]

(24) cannot be solved in its current form. However, Steuer shows that it can be reduced to

\[ \max D(x) = \{ d_1(x), d_2(x), \ldots, d_r(x) \} \]

subject to \( x \in X \)

where \( d_j, j = 1, 2, \ldots, r \), represent the extreme rays of the reduced gradient cone as defined by the bounds \( m_k \) and \( u_k \) on the weights. The gradient cone can perhaps best be visualized (in objective space) by imagining a light source at the origin which shines on the efficient surface. Initially the cone of light illuminates the whole surface and the cone is narrowed down as interval bounds are specified for the weights.

It is this process of narrowing down the light cone which Steuer has developed into an interactive method. In terms of the above illustration, the cone of light initially illuminates the whole efficient surface. 2q+1 extreme solutions dispersed over this illuminated area are presented to the DM who chooses the most preferred one. The central axis of the cone is then moved to this chosen solution and the cone is reduced to \( 1/4 \)th the cross sectional volume, thereby illuminating a proportionately smaller area around the chosen point. A further 2q+1 efficient extreme point solutions are presented, and the process continues until the DM requires the generation of all efficient extreme point
solutions within the reduced light cone. At this stage, the set of efficient extreme point solutions should be of a sufficiently small size to be comprehensible to the DM. Mathematically, this procedure generates a new set of interval bounds at each iteration, with each reduced gradient cone being defined from a new set of critical weights.

As with the ZW method, the DM is required to make holistic comparisons among solutions, and it is assumed that the DM has a linear utility function since only extreme point solutions are ever generated. However, there is no guarantee that the method will converge even under the assumption of an ideal DM with a linear utility function (Zionts [1982, pp 4.21-4.25]).

The Tchebycheff norm approach is a variation of the above method. However, this formulation is capable of generating every efficient point. The formulation is given below.

Min \( y \)

subject to \( y \geq w_k(U_k - f_k(x)), k = 1, 2, \ldots, q \) \( (26) \)

\( x \in X \)

\( w_k \geq 0, k = 1, 2, \ldots, q. \)

A small set of dissimilar efficient solutions are randomly generated from within the gradient cone at each iteration. One solution is chosen by the DM and the cone is reduced around this solution. As the gradient cone shrinks down at each iteration, the sequence of solutions is expected to converge to the most preferred solution.

6.9 The Method of the Displaced Ideal and the Reference Point Approach

The method of the displaced ideal (Zeleny [1976]) is perhaps more based on empirical studies of decision making behaviour than the methods reviewed thus far. It is based on the concept that choice between alternatives may differ depending on the point of reference which is used. The obvious choice for a point of reference is the ideal solution, which can be displaced as some solutions are excluded from the efficient set. The method, which is interactive, is illustrated in Figure 2.
for two objectives.

The initial compromise set is the entire efficient set ABE which reduces to BD and finally to BC, with the ideal point being displaced each time. The method for determining the compromise set is based on a set of weighted distance metrics with respect to the ideal solution, i.e.,

$$
(27) \quad \left( \sum_{k=1}^{Q} w_k (f_k - U_k)^p \right)^{1/p}, \quad 1 \leq p \leq \infty.
$$

Wierzbicki (1980) makes use of this concept of a displaced ideal in his reference point approach where it is assumed that the DM has certain goals or aspiration levels which are to be attained. The methodology is simple. After being exposed to the extreme solution matrix, the DM specifies a reference point or desired solution. The optimization process is to determine whether or not such a point is in fact attainable, and to present to the DM the efficient point which results from the optimization. If the sequence of reference points giving rise to a sequence of attainable points, converges, then the limit is the solution to the MOMP. At each attainable point information is given to the DM to aid in the choice of the next reference point. The
process of moving from a reference point $f^R$ to an attainable point $f^A$ is achieved using a scalarizing function $s(f - f^R)$. The simplest form of $s$ is that of a distance metric, not unlike that used by Zeleny for determining the compromise set.

The distinction between the two methods is that in Zeleny's approach the compromise set is determined, at least in part, by the weights which are placed on each objective. In the reference point approach the weights are implicitly incorporated into the reference point $f^R$ which is specified by the DM. The reference point approach has come under considerable study by Wierzbicki and his colleagues at IIASA (Lewandowski and Grauer [1982]). This has resulted in a number of practical applications of the method and the production of an interactive package entitled DIDASS which implements the method.

While the previous sections do not cover all interactive solution methods, the major concepts have been covered.

7. METHODS WITH A POSTERIORI ARTICULATION OF PREFERENCES

This final group of solution methods to be reviewed, where the DM provides information after solution, were among the first to be developed and deal with linear MOMP's. These methods which generate all efficient extreme point solutions (using the vector maximum approach of Section 2.2) were, at least initially, a theoretical development with little thought given to the role of the DM in the decision making process. Such an approach eliminates any methodological subjectivity by simply generating all efficient extreme point solutions for the DM to examine. The disadvantages lie with the size of the set of efficient extreme point solutions which should be large for any MOMP of non-trivial size.

The basic solution strategy for these methods is to begin with an efficient solution. Subproblem tests are then used to determine adjacent efficient extreme point solutions. A bookkeeping structure is maintained to ensure that all adjacent points to a given extreme point are identified and that each extreme point is visited only once. The process terminates when there are no extreme points which have not been examined. The computational requirements are high for other than small problems.
The methods of Evans and Steuer [1973], Yu and Zeleny [1973] and Ecker and Kouada [1978] are capable of finding all efficient extreme point solutions, with the Ecker and Kouada method simplifying the subproblem tests at each step to only a few pivots. Isermann [1977] and Ecker, Hegner and Kouada [1980] provide methods which are capable of generating every efficient solution. Their methods generate the set of all maximal efficient faces which are defined by convex combinations of their extreme points. Yu and Zeleny [1973] also propose an approach for calculating these efficient faces once the efficient extreme point solutions are known. However their approach is impractical for all but the smallest problems, since it implicitly enumerates all possible combinations of the extreme points, while solving a subproblem for each combination.

It is, however, reasonable to assume that the DM will only be interested in a subset of the set of efficient extreme point solutions. On the basis of this assumption, Ecker and Shoemaker [1981] reduce the size of the efficient set analytically by defining "types" of solutions deemed to be more desirable than others. Morse [1980] and Torn [1980] demonstrate the use of clustering techniques to group the set of efficient extreme point solutions into various types. A dendogram, which is a hierarchical agglomeration from individual solutions into one final solution type using the form of a tree diagram, is used to provide valuable information to the DM on the structure of the efficient set.

In another approach, which also presumes that the set of efficient extreme point solutions has been generated, Levine and Pomerol [1984] demonstrate the use of an interactive method entitled PRIAM. This approach, which is based on artificial intelligence methods, helps the DM explore the efficient set and find a most preferred solution.

Given adequate computing power, these methods would seem to be well suited to reasonably small MOMP problems, especially where some solutions are implicitly eliminated as in the Ecker and Shoemaker approach.

8. DISCUSSION

The various solution methods for the MOMP with a single DM reviewed above all require value judgements
(i.e., subjective input) from the DM. Therefore the first observation is concerned with the necessary subjectivity of all MOMP solution methods. Value judgements are necessary. In the majority of solution methods these judgements are progressively articulated until a most preferred solution is found.

As solution methods have been developed there has been a noticeable move away from requiring too much of a DM, both in the quantity and quality of information elicited. This is indicative of a trend toward increasing realism and is borne out by the examples of Sakawa and Seo (1983) and Stewart (1984) where the assumption of an ideal DM who always acts in accord with his or her utility function has been relaxed by using fuzzy sets or a probabilistic approach. Also, with the advent of computer graphics, information can be presented to the DM in a much more comprehensible form (e.g., Winkels [1982] and Ho [1985]). This, then, has been one of the major trends in the development of MOMP solution methods, namely, the adoption of a more realistic stance, achieved by reducing the requirements placed on the DM, and thereby making his or her participation in the decision making process as easy as is possible.

Different DM's will prefer different solution methods (see Buchanan [1985]). Even so, what is required from this point is that appropriate use be made of the large resource of solution methods which is available. This will involve an appreciation of both the decision making behaviour of the DM and the environment within which these decisions will be made.

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