AN APPROACH TO THE CREW SCHEDULING PROBLEM

G.R. Edwards
University of Auckland

SUMMARY

Minimising the cost of public transport operators has been posed as a set-partitioning I.P., where the variables are constructed to ensure the union regulations are satisfied and the I.P. constraints represent the timetable requirements. For bus crew scheduling realistic problems give rise to I.P. problems of excessive size. Using attributes for 'good' rosters, this paper shows how the number of variables can be greatly reduced enabling the optimal solution to an overconstrained I.P. sub-problem to be found in reasonable time.

1. INTRODUCTION

This paper is divided into four sections. In the first two the vehicle and crew scheduling literature is reviewed. The area is lacking in published papers, perhaps because much of the work carried out by commercial interests is confidential. In what has appeared it is surprising that research in particular fields, notably bus and airline scheduling has proceeded largely independently. The interrelation of these fields will be emphasized during the course of this review. In the third section a unified methodology for optimisation of the scheduling problem is developed with particular reference to the operations of a typical N.Z. metropolitan bus operator. Finally, some preliminary results are presented.

2. LITERATURE SURVEY

The objectives of public passenger transport are not immediately obvious. A Government white paper proposed the objective: To ensure that all sectors of the community enjoy an acceptable degree of mobility. For local authorities the objective is related to the welfare of the local area. Such an objective can be expressed as: to provide the necessary service at minimum cost or conversely to maximise service subject to budgetary constraint. For a private operator the objective is to maximise profit or minimise cost. From an operational standpoint however these problems coalesce.

Four components of the Scheduling problem have been identified [26]: (a) Timetable construction - characterisa-

*Manuscript submitted June 1979, revised October 1979
tion of all bus trips; (b) bus scheduling - assignment of trips to vehicles; (c) crew scheduling - crews allocated to the bus schedule in compliance with their labour award; and (d) duty rostering - compilation of daily shifts into weekly/monthly rosters with allowance for cyclical patterns of work and rest days. It is clear that the airline scheduling problem is directly analogous to the bus scheduling problem. Many authors have noted the desirability of solving the problem as a whole, rather than in stepwise fashion, while admitting it would be computationally infeasible to do so [26][38][24]. We shall now review in detail these components.

(a) Timetabling: Programs have been written in the U.S. to accept as data the counts of passengers in each small period of the day and to generate automatically the bus trips, i.e., the timetable [6, 8]. In Britain, transport authorities commonly specify headways (service frequencies) based on local demand and other operating considerations, thus defining the timetable [26]. In other countries, including N.Z., the timetable evolves in response to demand within limitations imposed by historical route structures and local conventions. It is thus defined by fixed route structure and departure times.

(b) Bus Scheduling: Where a timetable is specified by headway parameters in addition to a set of specified trips, there is, in principle, interaction between components (a) and (b) of the scheduling problem. This interaction is emphasized in welfare maximisation models where the timetable and bus schedule are constructed simultaneously to maximise a service level within budgetary constraints [2]. Wren [26, 37] et al., have developed a suite of heuristic programs (TASC and VAMPIRES) to solve related models. Wren's basic algorithm [35] attempts to minimise the number of buses used plus a function to account for running costs. The program will produce a 'lower bound' to the number of vehicles or this may be specified together with service parameters, essential trip data and inter-terminal running times. An initial solution is found which is revised by a series of improvement routines. The final solution may or may not contain infeasibilities. If unacceptable infeasibilities occur they can be removed by retiming trips or rerunning the program with an additional vehicle. This system incorporated in the VAMPIRES package has been modified several times and, in particular, includes an interactive facility [36]. Wren justifies this heuristic procedure on the basis of favourable comparison with other methods and manual solutions.

In a more rigorous manner the problem of scheduling buses by linking pairs of trips has been posed as a standard assignment problem [18]. A square table is formed with a row for every trip arrival and a column for every trip departure. The (i,j)th table entry is the cost of using the vehicle from trip i to complete trip j. This cost will be
a function of the distance, time and running costs between arrival and departure and will be arbitrarily large for infeasible connections. The objective is to find the row and column allocations which minimise the total cost. The number of assignments to infeasible connections represents the number of buses required since no suitable matching arrivals can be found. In any practical problem the matrix becomes very large and only costs dependent on trip linkages can be considered. Wren's heuristics \cite{35, 36} using improvement routines which examine certain trip linkages to see if the overall solution is improved could be considered a heuristic approach to this assignment problem.

A shortest path algorithm is incorporated in the RUCUS bus and crew scheduling package, described by Bennington and Reibo \cite{6}, to find the shortest paths through a network of nodes representing bus trips. Due to problems of size the program severely restricts the number of links formed by any one node and only considers a single depot. The artificiality of the model has been criticised and the heuristics are not generally appropriate outside the U.S. situations for which it was developed.

Graph theory has been applied independently, although in a very similar manner by Saha \cite{29} to the bus scheduling problem and by Levin \cite{20} to aircraft fleet scheduling. A directed graph $G = [N; A]$ is defined as a collection of $N$ elements (bus trips) together with a subset, $A$ of ordered pairs $(x, y)$, where $x, y \in N$. The members of $N$ are called nodes and the members of $A$ arcs (trip linkages). When flows are associated with the arcs the graph is referred to as a network. A sequence of nodes and arcs $x_1, (x_1, x_2), \ldots, (x_{n-1}, x_n)$ is defined to be a directed chain for $(x_i, x_{i+1}) \in A$, $\forall i=1, \ldots, n-1$. If $x_1 = x_n$, $n > 2$, then the sequence is a cycle. A directed graph that contains no cycle is said to be an acyclic directed graph. A decomposition of an acyclic graph is a partition of $N$ such that the nodes of each part with their connecting arcs, form a single chain. The route travelled by a bus or aircraft (i.e., the set of successive trips or flight legs) is a chain. The planning period is taken to correspond to the minimum repeating unit of the timetable, e.g., weekly. The corresponding graph is acyclic because time progresses in acyclic fashion within the planning period. Consequently, the minimal fleet size problem can be restated as that of finding the minimum number of chains into which the given acyclic graph can be decomposed. By constructing a bipartite graph $G^* = [S, T; A^*]$ where $\{S\}$ and $\{T\}$ represent origins and destinations at specific times, from $G$, and assigning a flow of one to all valid arcs $\in A^*$ (i.e., valid trip linkages), Levin has shown that the problem of determining the minimal number of chains into which $G$ can be decomposed can be solved by finding the maximum flow in $G^*$. Fulkerson's out of kilter algorithm \cite{10} could be used to efficiently solve the resultant maximum flow problem.

One can assume that a bus arriving at a destination will
depart from this point some time later. Alternatively, the bus may be allowed to 'deadrun' to some other point before commencing a timetabled trip. In this case the graph $G^*$ will include arcs derived from nodes corresponding to trips arriving at one point into nodes representing trips departing from some other point. For dead running to occur, such an arc must be included in a chain. Inclusion of such 'dead running arcs' (c.f., Saha's comment 'can a new set of trips be added so that the minimum number of buses be reduced?' [30]) in the general case should not prove difficult. For example, all 'deadrunning arcs' that involve less than specified maximum distances and times could be included in the graph $G^*$. Care would have to be taken to ensure the resulting problem size did not become excessive. This possible provision for 'deadrunning' seems to have eluded both Saha [30] and Wren [34] in their respective comments on Saha's original paper.

Despite the apparent attraction of this naturally integer method it relies on the assumption that bus routes (chains) can be compiled merely by considering the feasibility of the links (arcs) independently. Thus it cannot be applied to vehicle scheduling problems where the chains have overall capacity and distance constraints or to crew scheduling problems where award conditions apply to the whole chain, i.e., where inclusion of an extra arc and node in the chain depends on the nodes and arcs already included.

Wren comments that Saha's results closely resemble an elementary queueing theory approach to bus allocation. In similar vein, a Wellington study [31] using LIFO queuing from a central dispatch point achieved some savings over the existing manual procedure. This illustrates that care must be taken to ensure that the complexity of a method, particularly a heuristic does not obscure any simple relationships involved.

(c) Crew Scheduling: Early attempts to automate bus crew scheduling were not encouraging [18]. Young and Wilkinson [38] concluded that due to the complexity of the constraints the problem probably could not be solved by computer. Elias [8, 9] suggested a method where the large number of possible duties were listed in ascending order of cost. A roster is constructed by selecting for a particular trip, the least-cost duty that contains that trip. All other trips performed by that duty are deleted from the timetable and a new duty is selected similarly for the next trip not yet covered. This process is continued until the timetable is covered. A roster is chosen that has the least total cost among a set of rosters, obtained as above, by starting with each run in the timetable. Clearly, this gives rise to a roster composed of good and very bad duties which is subsequently refined by improving the worst duties that it contains. The method was found to be computationally expensive.

The Municipal Tramways Trust, Adelaide [15], in association with Bennett and Potts [4],[5] has developed a series
of algorithms for the solution of the bus crew scheduling problem for a given bus timetable. The overall procedure, while requiring some final manual adjustment, did produce rosters comparable to those obtained manually, considerably more quickly and was considered a valuable management tool. Unfortunately, due to its heuristic nature there is no obvious generalisation of this method. The method is divided into three phases. Phase I is a heuristic, essentially automating manual procedures, to subdivide the given timetable into 'pieces of work' to be sequentially included in the roster until the timetable is covered. It was considered impossible to devise a mathematical technique that would ensure optimal results. Phase II operates on the portions of work output from Phase I that do not constitute straight duties. The Hitchcock problem algorithm of Ford and Fulkerson [10] is utilized to optimally pair these portions of work to form broken duties. This procedure is only optimal in the sense that it constructs optimal broken duties from certain parts of the timetable as specified by the Phase I heuristic. The third phase is briefly discussed under duty rostering (d).

Wren et al [26, 35, 36] has developed a suite of heuristic bus crew scheduling programs with the bus schedule and the labour award as input data. The approach consists of constructing tentative duties according to a set of parameters specifying the numbers and types of duty and other general features. Improvement routines are used to try to improve on an initial schedule which satisfies all the constraints. To permit general application, specially written subroutines are used to accommodate any unique requirements of a particular undertaking. The RUCUS package contains heuristic procedures described by Wilhelm [32]. Other reviewers have described the results of these methods as disappointing, considering the attention given to them [24]. This is in part due to the difficulty in adapting heuristic procedures developed in a particular situation for general use.

With respect to the airline scheduling problem, the basic planning unit is the rotation, generally implying a round trip taking a crew from home base and returning there at the end of the journey. Mathematically, air crew scheduling is a weighted set covering problem [2]. The objective is to select a set of rotations such that each flight leg is covered at least once and the total cost is minimised. When rotations are chosen such that a leg is covered more than required, dead heading occurs, i.e., crew occupy revenue seats in order to position themselves to home base or to the next flight. Each rotation must satisfy union, company and safety regulations. A rotation can be represented by a column of a matrix $A = \{a_{ij}\}$. The rows represent legs. If leg $i$ is included in rotation $j, a_{ij} = 1$, otherwise 0. Airlines commonly formulate the scheduling problem as follows: Given $A = \{a_{ij}\}$, and $j \in \{J\}$, the set of all duties or ro-

...
tations,

\[
\text{Minimise } z = \sum_{j=1}^{n} c_j x_j
\]

subject to \( \sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad i=1 \ldots m \)

where \( x_j = 0,1 \) for exclusion or inclusion of rotation \( j \) respectively, \( c_j \) represents the cost of choosing rotation \( j \), \( b_i \) represents the number of crews allocated to leg \( i \) and there are \( m \) flight legs to be covered.

Because of the combinatorial nature of the problem, in practice, the number of legal rotations becomes unmanageably large. In a review of procedures used by major airlines, Arabeyre et al. [2] found that generally the number of possible rotations was reduced heuristically. One of a variety of integer linear programming or heuristic methods was then used to solve the reduced problem as follows: (i) Martins Euclidean Algorithm [33] (a cutting plane method which is claimed to be faster than Gomory's original code); (ii) A variety of branch and bound methods, some utilizing special features of the problem, e.g., cost projection whereby ordering the matrix columns with respect to cost per leg eliminates the need to wait until a partial solution actually exceeds the current bound; (iii) A set covering algorithm due to House et al [17]; (iv) A combinatorial programming method based on work by Pierce [27], which has close analogies to the set covering algorithm of Garfinkel and Nemhauser [13]; (v) Heuristics. In summary, the airlines solve the crew rostering problem heuristically, at least to the point where various efficient integer programming algorithms are able to cope. Arabeyre et al. [2] conclude although considerable progress has been made no 'good' solution has been found to the general problem.

Rubin [28] notes that in many cases the constraint matrix \( A \) cannot be generated much less solved without imposition of severe ad hoc limitations. He proposes the use of a set covering algorithm repeatedly on smaller matrices extracted from the overall problem. He claims this heuristic approach, essentially approximating the global optimum with a sum of sub-problem optima, produces significant improvement to air crew schedules where the data is too vast to be handled manually.

More recently Gerbacht [14] states that although crew scheduling programs reliably produce rosters comparable to those produced manually, it is still necessary to improve an existing solution by a series of steps rather than to exhaustively solve the original problem. A new integer programming algorithm utilizing surrogate Lagrangian constraints is presented which is much faster than previously reported methods [22]. Larger problems must still be partitioned into a
series of smaller subproblems in order to keep computing requirements within reasonable limits however.

Although linear and integer programming models have generally been discarded for bus scheduling, air crew scheduling research has concentrated on finding better solution techniques for the set partitioning model, recognising the problems of size. The air crew methods adopted could be termed 'good' or 'optimal' solutions to overconstrained problems rather than pure heuristics.

In the area of bus scheduling one exception to the general pattern is the work of Mason [23] who proposed, without reference to the aircrew scheduling literature, a set partitioning model, directly analogous to the model presented by Arabeyre et al. A collection of heuristics are presented to restrict the number of variables (i.e., feasible duties) to a manageable level in a similar manner to Elias [8, 9]. However his model does not include provision for broken shifts and there remain doubts about the computational viability of the method. The physical problem presented is small and even after some manual manipulation the final solution is infeasible and more costly than the actual manually prepared roster.

(d) Duty Rostering: Duty rostering involves the aggregation of daily shifts into weekly or monthly rosters. This problem is addressed in Phase III of the Adelaide study [15] which the authors have subdivided as follows: (i) construct an optimal day off pattern; (ii) limit overtime to a minimum; and (iii) maximise the number of identical or similar start time schedules without increasing overtime. The algorithm is a heuristic operating on Phase I and II output, utilizing integer programming and Hungarian assignment algorithms to solve various subproblems generated by the heuristic. It has been noted that such a model could not be generalised for application to the air crew rostering problem [25].

Nicoletti [25] poses the air crew rostering problem as a modified set covering problem, actually an assignment problem, in similar vein to Agard [1]. He shows how the method of Ford-Fulkerson for maximal flow over a capacitated network is more efficient than the Hungarian method, such as that proposed by Agard to automate monthly planning for airline stewards. Due to the complexities and subtleties of local arrangement, seniority priorities and other customs it is generally felt that automation of duty rostering offers the least promise and benefit.

We can summarize these contributions as follows: It is possible to construct optimizing models which will give minimum cost vehicle and crew schedules, but due to their size their solution is beyond present computer technology. Most mechanical solutions will either be heuristic or chosen from an over-constrained subproblem. The methods can be classified as:

(a) Optimisation of overconstrained subproblems, using (i) network approaches, (shortest path or maximal flow algorithms which do not allow overall
constraints), and (ii) integer programming (massive set covering and set partitioning algorithms which can handle any constraints).

(b) Heuristic solutions, some of which utilize optimising algorithms at specific stages of the process.

The major difficulties with heuristic approaches are the isolation of the assumptions inherent in the often complex heuristics and the lack of any measure of how good the solution is. However, these methods may be faster and will be closer to manual procedures, in contrast to the optimisation of over-constrained subproblems. The greater the number of overconstraints expressing restrictive premises or assumptions, the smaller the resulting model. However, the solution will only be valid if these assumptions are realistic and justifiable. It is thus necessary to balance the restrictive nature of the overconstraints against the consequent size of the model.

3. THE GENERAL VEHICLE SCHEDULING PROBLEM

The purpose of this brief discussion of vehicle scheduling is to illustrate its common characteristics with the bus and aircraft scheduling problems reviewed earlier. Later a general approach will be formulated that will have application to both.

The common feature of the bus, vehicle and crew scheduling problems is that they can be posed as set-partitioning problems with increasing degrees of additional structure,

\[
\begin{align*}
\text{minimise} \quad & C^T x \\
\text{subject to} \quad & A x = b
\end{align*}
\]

where \( b_i \) is an integer, \( x_i = 0,1 \) according to whether chain \( i \) is included in the partition; \( A (=a_{ij}) \) \( i = 1, \ldots, m; \)
\( j = 1, \ldots, n \), where \( a_{ij} = 0,1 \) according to whether trip \( i \) is included in chain \( j \); \( m \) is the number of trips and \( n \) the number of chains; \( C_j = V + m_j \) with \( V \) the mileage equivalent cost of each vehicle and \( m_j \) the total mileage of route \( j \).

The simplest form where the objective coefficients are all unity and \( b \) vector is a column of ones, corresponds to the minimal fleet size problem, commonly formulated as a network flow model. The only restriction on the construction of the chains is that any two consecutive non-zero elements in a column (or chain) represent a feasible link. In the bus scheduling problem where running and capital costs are taken into account the objective coefficient vector is a more descriptive cost vector \( c \) reflecting the running and capital costs. The more general vehicle scheduling problem has in addition to the feasible link condition the requirement that the chains or columns satisfy overall distance and capacity constraints. The crew scheduling problem has analogous but generally more complex overall restrictions comprising the award, safety and other operational constraints. Air crew scheduling has in addition the possibility of a more general
integer RHS vector b, instead of unity vector e, representing multiple crew requirements.

4. A BUS AND CREW SCHEDULING MODEL

I shall first outline the bus operations typical of the main cities in N.Z. which contrast in some respects with the U.K. and North American systems commonly addressed in the literature. In a typical N.Z. operation the labour costs comprise at least 2/3 of the total cost of running public transport. Thus making efficient use of driver time is of prime concern [3]. Evolution of the present radial route structure is historical and may no longer be appropriate, particularly in off-peak periods. This is due to the increase in private mobility, growth of suburban shopping areas and general peripheral development. The consequent decrease in off-peak travel has resulted in rapidly escalating transport losses. Various options have been proposed [21, 7, 12]: (a) a reduction in service; (b) increase in service; (c) a more general route network e.g. incorporation of circular routes; and (d) a change in emphasis from a network of fixed route to a community focused flexible network of transport centres with feeder services to activity centres, i.e., nodal system. It is clear that a crew and vehicle scheduling tool is required by management to investigate the cost structures of these options.

In N.Z. at present, a timetable is constructed as a set of trips compiled with the aid of statistical surveys of demand and logistical factors, such as running time, given a historical structure. Within this limitation the transport authorities attempt to satisfy transport demands between specific terminals at specific times. The timetable is characterised by uni-directional peak demands in the morning and evening, which are of the order of 3 times off-peak demands. Only a relatively small number of off-peak trips are inserted to maintain a minimum service throughout the day. Due to the decline in patronage particularly in off-peak periods it is considered important that scheduled services cater for a demand rather than to provide a level of service. Hence the scope for timetable perturbation as a consequence of scheduling considerations is limited, i.e., parts (a) and (b) of the Scheduling problem can be considered independent, although the identification of trips costly in the scheduling sense would be valuable.

N.Z. services are manned entirely by one man buses and relief and meal break points only occur at a few specific depots. In general terms labour awards seem comparable worldwide, except in N.Z. there is a restriction on the proportion of broken shifts in a given roster. This one to one correspondence of bus and driver means that parts (b) and (c) of the Scheduling problem can be combined. The bus or driver performing a given task need not be specified but each task must
be feasible with respect to the timetable, logistics and the labour award. Objective coefficients would include labour, running and fixed cost components. The fleet size is determined by the peak number of buses required and hence the capital, fixed and total running costs are independent of the allocation of buses to drivers or duties. Local knowledge and unwritten conventions mean that the rostering stage is easily handled manually as has been observed elsewhere [26]. This is in contrast to the Adelaide study [15], where as a consequence of heuristic crew scheduling, labour costs are included at the rostering stage.

Ignoring the statistical analysis and market research necessary for timetable construction and accepting the present manual rostering procedure, results in a simplification of the scheduling problem in the N.Z. context to simultaneous solution of parts (b) and (c). This can be stated as: scheduling bus/drivers to operate a given daily timetable ensuring that the driver duties satisfy the union regulations at a minimum total cost to the transport authority.

Consider again the model given by expressions (2). Given its formidable size, the approach presented here is to solve optimally an overconstrained model. Imposition of additional physically reasonable structure will give rise to a greatly reduced feasible region, containing at least excellent sub-optimal solutions to the original problem. This is to be compared with a sub-optimal solution of a model constrained by complex heuristics as is generally the case in the literature. Justification for such a procedure relies on the "common sense" of the additional restrictions; the favourable comparison with manual/alternative mechanical solutions for selected problems; and favourable comparison with respect to computer resource requirements with alternative methods or reasonable limits. The basis of this procedure is a generalisation and extension of the petal concept developed by Foster and Ryan [11] for the vehicle scheduling problem.

(a) The Objective function:

The formulation assumes that the bus operator's objective is the minimisation of total operating costs. For a typical N.Z. metropolitan bus operator in 1979 these costs amount to approximately:

- Cost of ordinary time driver claim hour $4.00
- Running cost per kilometre $0.20
- Fixed cost of bus ownership per day $45.00
- Initial cost for driver training and equipment $4200.00

This results in the following objective function:

\[
\text{Min } f = \min \{ \sum_{j=1}^{n} [4.00(Hw_j + Hi_j) + 2.00 Ow_j + C] \}
\]
where $H_{wj} =$ working (i.e., actual driving) hours contained in duty $j$;
$H_{ij} =$ idle time hours contained in duty $j$;
$O_{wj} =$ overtime hours contained in duty $j$ (an extra half-time payment in addition to ordinary time is incurred for such hours);
$K_{rj} =$ route kilometres travelled on timetabled trips on duty $j$;
$K_{dj} =$ deadrunning kilometres of duty $j$;
$C =$ cost to reflect initial training and equipment outlay;
$n =$ total number of duties to cover the timetable.

Objective function (3) can be rearranged as

$$f = \sum_{j=1}^{n} [C + 2.00 \times O_{wj} + 4.00 \times H_{ij} + 0.20 \times K_{dj}]$$

where $\varphi(T) =$ only a function of the timetable prescribed and can thus be regarded as a constant and neglected in the minimisation; The total claim hours are represented by $\sum_{j=1}^{n} [H_{wj} + O_{wj} + H_{ij}]$; $n$ is the total number of drivers required since each duty represents a one-man shift; and the total kilometres travelled is given by $\sum_{j=1}^{n} [K_{rj} + K_{dj}]$.

The cost $C$ must reflect the relative importance of the initial costs of driver hire and training. A high cost for $C$ will tend to decrease $n$, largely at the expense of increasing overtime payment and deadrunning. $n$ is also indirectly related to the peak number of buses. A reasonable but arbitrary cost for $C$ on a daily basis could be determined as follows:

$$C = (\% \text{ staff turnover} \times $4200)/(365 \times 100).$$

This neglects the tendency of a minimum $n$ to give a minimum number of peak buses. If it is found that too large a number of peak buses is required then $C$ should be increased accordingly.

The following is a more direct although approximate method of obtaining an objective function coefficient to minimise the peak bus requirement. The peak number of buses is determined by the maximum number of bus routes and specials (i.e., deadrunning or idle) active at any one instant of the day. This parameter cannot be determined exactly by treating each duty independently. Exact calculation required specification of a complete roster and timetable. However a first approximation can be deduced from the timetable alone, as the maximum number of trips active at any one instant of the day. Having identified the time at which this peak demand occurs let us define a variable $p_{j} = 0,1$ according to whether duty $j$ includes active duty (either passenger-carrying, idle or deadrunning) at this time. Then $\sum_{j=1}^{n} p_{j}$ would be an approxi-
measure of the peak bus requirement. The fleet size is determined by the peak bus requirement at any time throughout the minimum repeating unit of the timetable, normally a week. If any day(s) had a characteristically higher peak bus requirement the peak bus coefficient would be zero for all other days. With these considerations the final objective function becomes

\[
\min f = \sum_{j=1}^{n} \left[ 4.00 H_i + 0.20 K_d + 2.00 O_w \right] + \frac{n \times \% \text{ staff turnover} \times \$4200.00}{365 \times 100} \]

\[ + \sum_{j=1}^{n} 45.00 p_j.\]

Other models have not used such a general objective function. Bennett and Potts [5] only explicitly consider overtime, and Mason [23] while using a similar form ignores the latter two terms and uses an incorrect cost coefficient for the overtime component. Only the marginal cost of overtime relative to ordinary time need be considered since \( \phi(T) \) already includes the ordinary time cost. Wren [26] considers all but initial driver cost, although in stepwise rather than simultaneous fashion. The objective functions in [26] minimise idle time and empty running, rather than overall cost.

(b) Pre-analysis: The first size reduction heuristics consist of an analysis of the timetable input data. Pictorial representation of such data is surprisingly informative. To this end the timetable is output in graphical form (e.g., trips active vs time) enabling identification of peak demands and subsequently the periods of the day where broken shifts may be appropriate.

The maximum number of trips active at any one instant will give a lower bound to the peak number of buses required. Comparison with a similar graph for the final roster would indicate whether deadrunning or idle time has resulted in an unacceptable increase in the peak bus requirement. This representation may also indicate trips which result in bottlenecks. Such information could be used to pinpoint areas suitable for timetable perturbation.

Examination of the individual trips may indicate that certain pairs of trips can be classified as being naturally "out and back" and can thus be combined. Other circumstances such as local custom or overwhelming logistical evidence that two or more trips should always be performed sequentially may allow further aggregation of trips. Any reduction in the number of trips will result in a substantial decrease in the numbers of variables particularly and constraints and hence
in the size of the problem.

(c) Overconstrained Duty Construction: The union regulations for a N.Z. metropolitan bus operation can be briefly summarised as follows: A driver is allowed 20 minutes sign-on and 10 minutes sign-off time within the 8 hours to 9 hours 10 minutes maximum permitted range for straight duty total sign-on time. Time worked in excess of 8 hours is paid at time-and-a-half and a meal of at least 30 minutes duration must be provided after 2 hours 30 minutes and before 4 hours 45 minutes elapsed sign-on time. For broken duties, each portion must consist of 2 to 6 hours work separated by a minimum 2 hour break. The maximum spread from the first sign-on time to the second sign-off is 12 hours, but the actual signed-on time range permitted is as for straight duties.

The duty sets constructed below have been developed with the aim of formulating the smallest problem that remains a reasonable approximation to the real life situation. All columns of the A matrix (i.e., the IP variables representing possible duties are constructed so that logistical restrictions and union regulations are satisfied, including those applicable to the duty as a whole. In the bus/crew scheduling problem the petal set of duties) will have the additional property that any two consecutive trips performed in the same duty will be such that the second is the next possible that can be worked after the first. This condition restricts the petal duties to the straight duty category. The imposition of this unique ordering in time on duty structure is to be compared with the geographical ordering of Foster and Ryan [11] in the vehicle scheduling problem who used a travelling salesman algorithm to combine radially continuous locations. In the bus/crew scheduling problem the time ordering is deduced by a simple rule directly from the timetable and input data.

On account of the meal break requirement and the provision for broken shifts, the following modified petal set is useful. The modified petal set satisfies the same conditions as the petal set except that the next possible condition for consecutively worked trips may be relaxed for the purpose of including a meal break or inserting the minimum 2 hour break between portions of broken duties. A relaxed petal set comprises duties that are more general perturbations of the petal set than are contained in the modified petals. The first relaxed set is constructed as for the modified petals except that the next possible condition is further relaxed to include the next possible run but one. The jth relaxed set has the properties of the modified petal set except that for any consecutively worked trips the second must be at most the (j+1)th trip that can possibly be worked after the first. These relaxed sets are termed complete since the relaxed conditions apply to all trips in the timetable.

Selected relaxations refer to duty sets where further secondary conditions are imposed by means of a set of parameters. The parameters are chosen in such a way as to
exclude as many as possible of the duties that are unlikely to be found in a good solution. These parameters are:

(i) COST - A relaxed duty must have an objective cost less than a specified maximum COST. Although a high cost duty may allow inclusion of many low cost duties in a 'good' basis, such an unbalanced roster is unlikely to be acceptable.

(ii) PACKING - A relaxed duty must consist of at least a specified minimum number of trips (PACKING). It is sensible particularly from an industrial relations standpoint to have duties with a largely uniform work content. Of course a highly packed duty tends to have a low objective cost.

(iii) TIME - It is unnecessary to generate straight duties starting later than a specified TIME as they are unlikely to constitute a reasonable day's work. COST and PACKING parameters if chosen harshly will have this effect also. It is only necessary to construct broken duty portions to cover peak periods as specified by TIME.

(iv) TRIP - the petal conditions may only be relaxed for specified TRIPS. The specified trips could be deduced from statistical data for columns generated already or from shadow price data provided by the current optimum LP solution or possibly by direct analysis of the timetable. In this way further duties can be constructed to contain trips, recognised as difficult to include in the roster, or where there are relatively few duties containing particular trips.

The preceding classification of duty sets can be considered as an ordering of the A matrix depicted in Figure 1. Typically the constraint matrix (A = (a_{mn}), where n>>m and a_{ij} = 0,1 according to whether trip i is performed by duty j) is of this shape. The ordering is such that, moving from left to right, duties increasingly deviate from petal form. Since a large deviation from petal form generally results in a high objective cost the discarded portion is unlikely to contain 'good' duties. The other partitions indicated are convenient in terms of duty generation and eventual LP solution.

The objective of the bus/crew scheduling problem is to minimise a weighted sum of the number of buses and drivers and unproductive running and paid time, within logistical, timetable and award restrictions. In general terms this is achieved by a roster of approximately uniform duties containing as little unproductive time and performing as many
FIGURE 1: Diagramatic Representation of m x n Constraint Matrix

trips as possible. The petal concept and its modifications and relaxations is consistent with this aim. A petal or closely petal duty will contain relatively little unproductive running or time, will tend to maximise the average number of trips per duty and consequently minimize the number of buses and drivers. Examination of manual solutions reveals that most if not all manually constructed duties are of petal form or small perturbations thereof.

(d) Solution: It will be shown in a subsequent paper that, if the LP resulting from removal of the formal integer conditions is solved over petal or nearly petal duty sets, the solution is likely to be naturally integer. This results from the high proportion of integer basic feasible solutions constructed from such duty sets. Due to the likely proximity of integer bases to a given fractional basis conventional branch and bound techniques will rapidly produce an integer solution of similar objective contribution.

Two general solution strategies are suggested:

(1) (i) Generate modified petal set.
(ii) Solve LP and store solution.
(iii) Generate a complete set of first relaxations.
(iv) Solve new LP using solution (ii) as an initial basic feasible solution.
(v) Examine column statistics to generate further selected relaxations.
(vi) Solve over new LP using previous optimum as an initial basic feasible solution, utilizing the mixed integer programming option if LP solution is non-integral.
(vii) Repeat (v) and (vi) if necessary.

(II) As for strategy (I) except that after step (iv) examine the LP solution and remove the best duties. Define a sub-problem with a timetable consisting of those trips not covered by these best duties. Repeat steps (i) to (iv) for the sub-problem and either add the solution to the stored best duties or resolve over all LP columns generated in steps (i) and (iv) for both the sub-problem and the original problem.

Solutions of (I) and (II) are optimal within the given assumptions provided (II) is resolved using all LP columns generated. The addition of partial solutions in (II) however will provide a good solution 'bound' quickly.

Investigation of the model and solution method with actual data is only at a preliminary stage. However, the problem presented by Mason [23] referring to the A.R.A.'s North Shore Sunday timetable has been examined. Since this is a weekend timetable, split shifts are not permitted and the cost coefficient for the peak number of buses is zero. Some results are briefly summarised Table 1. An approximate scaling constant $\phi(T)$ is added so that the objective value plus $\phi(T)$ is proportional to the total cost of the roster.

<table>
<thead>
<tr>
<th>Variable (Duty) Set</th>
<th>$\phi(T)$ + Naturally Integer</th>
<th>B6700 Process Time (minutes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Petal</td>
<td>240</td>
<td>Yes</td>
</tr>
<tr>
<td>Modified Petal</td>
<td>195</td>
<td>Yes</td>
</tr>
<tr>
<td>First Relaxation</td>
<td>170</td>
<td>Yes</td>
</tr>
<tr>
<td>Selected further Relaxations</td>
<td>160</td>
<td>No</td>
</tr>
<tr>
<td>Mason (infeasible)</td>
<td>141</td>
<td>-</td>
</tr>
<tr>
<td>Problem 'Bound' (as in (II) above)</td>
<td>134</td>
<td>Yes</td>
</tr>
<tr>
<td>Manual</td>
<td>131</td>
<td>Several weeks</td>
</tr>
</tbody>
</table>

It should be emphasized that these preliminary results have not involved a full investigation of the effects of the generation parameters described earlier. At this stage the aim was to show the efficacy of the method. However, we believe that with tuning of the algorithm a better solution than the manual one can be found.

The Burroughs Mathematical Programming System TEMPO was used as the only readily available LP package. The inability to interact with TEMPO after each or any iteration is costly in terms of computing resources required. Typically a graph of objective value against time for set partitioning problems
is a step function. The plateaux represent the large number of variables which typically enter and leave at zero level before the objective value actually changes and finally a long period of pricing until optimality is verified. We believe that interaction with this process would allow solutions to be obtained more efficiently.

In conclusion a preliminary investigation of this overall approach to the bus/crew scheduling problem has demonstrated its viability. Further work must involve investigation of the effects of the various relaxation parameters and examination of the more general weekday problem requiring split shifts and peak bus contributions to the objective.

Acknowledgement

I would like to thank my PhD supervisor, Dr D.M. Ryan, for many helpful discussions and the DSIR for granting study leave to complete this research.

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