TIME MINIMIZATION IN TRANSPORTATION PROBLEMS

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Summary

This paper consists of two sections. A technique for minimizing time in a transportation problem is developed in Section I. Section II extends this methodology to multidimensional transportation problems having three indices. The procedures developed are finite and are based on moving from one basic solution to another until optimality is reached. Numerical examples illustrating the techniques are included.

1. Introduction

The cost-minimizing transportation problem, popularly known as Hitchcock-Koopmans problem, may be expressed as

\[
\text{minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} x_{ij}
\]

(1) subject to \( \sum_{j=1}^{n} x_{ij} = a_i, \) \( i = 1, 2, \ldots, m \)
\( \sum_{i=1}^{m} x_{ij} = b_j, \) \( j = 1, 2, \ldots, n \)
\( x_{ij} \geq 0 \)

For a feasible solution to exist it is necessary that total supply equals total demand. If in reality supply is greater than demand, a fictitious destination may be used to create the desired equality. If demand exceeds supply, a

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fictitious source is introduced.

Here the objective function is linear and convex. Algorithms, such as Dantzig's simplex method [2], provide efficient means of locating the optimal solution. But there may arise situations in which minimizing the time of transportation is of greater importance than cost, for example in the transportation of perishable goods, military equipment during war time, fire services, ambulance services, etc.

In a time minimizing transportation problem, a time matrix $\{t_{ij}\}$ is given where $t_{ij}$ is the time of transporting goods from $i$th origin to $j$th destination. For any feasible solution $X = \{x_{ij}\}$ satisfying (1), the time of transportation is the maximum of $t_{ij}$'s among the cells in which there are positive allocations, i.e. the time of transportation is $(\max t_{ij}: x_{ij} > 0)$. The aim is to minimize this quantity, i.e., find $x_{ij}$, $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$ that

$$\min [(\max t_{ij}: x_{ij} > 0)] \quad (\text{PROBLEM I})$$

subject to constraints (1). The time of transportation is independent of the amount of commodity shipped from suppliers to consumers while the cost of transportation is dependent on the variation in quantity of commodities shipped. Methods of minimizing transportation time for two-dimensional problems have been given in [3,4,6]. In Section 2 we present a new approach.

The multidimensional transportation problem with respect to cost is studied by Corban [1]. Mathematically, the cost minimizing three-dimensional transportation problem is

$$\min Z = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} c_{ijk} x_{ijk}$$

subject to
One economic interpretation of this problem is the following: $c_{ijk}$ is the cost of transporting a unit of product from origin $i$ to destination $j$ by means of transportation mode $k$; $x_{ijk}$ is the amount of product transferred from origin $i$ to destination $j$ using transportation mode $k$; $a_i$ is the supply available at the $i^{th}$ origin; $b_j$ is the amount required by $j^{th}$ destination, and $c_k$ is the total amount that can be transferred using the $k^{th}$ means of transportation.

The three-dimensional time-minimizing transportation problem can be stated as

$$\begin{align*}
\text{minimize} \quad & \text{Max} \sum_{i,j,k} t_{ij} : x_{ijk} > 0 \quad \text{(PROBLEM II)} \\
\text{subject to} \quad & (2), \text{where} \ t_{ij} \text{is the time of transportation of goods from} \ i^{th} \text{origin to} \ j^{th} \text{destination using transportation mode} \ k. \text{It is further noted that this time of transportation is independent of the amount of goods shipped as long as} \ x_{ijk} > 0, \text{where it is assumed that all carrier start simultaneously.}
\end{align*}$$

The three-dimensional time-minimizing transportation problem has been studied by Sharma & Swarup [7]. The method presented here is a direct extension of the approach proposed for the two-dimensional case.
2. The Two-dimensional Time Minimizing Transportation Problem

The two-dimensional time minimizing transportation problem was defined by PROBLEM I. The objective function of PROBLEM I is not convex. We need the following definitions.

(i) A feasible solution: A set \( X = \{x_{ij}\} \) of non-negative numbers satisfying (1) is called a feasible solution.

(ii) An improved feasible solution: Let \( X_1 = \{x_{ij}^1\} \) and \( X_2 = \{x_{ij}^2\} \) be two feasible solutions of PROBLEM I, and let \( M_1 = \{(i,j) : x_{ij}^1 > 0\} \); \( M_2 = \{(i,j) : x_{ij}^2 > 0\} \)

\[
T_1 = \max_{(i,j) \in M_1} t_{ij} \quad ; \quad T_2 = \max_{(i,j) \in M_2} t_{ij} \\
R_1 = \{(i,j) : t_{ij} = T_1, \ (i,j) \in M_1\} \\
R_2 = \{(i,j) : t_{ij} = T_2, \ (i,j) \in M_2\}
\]

\[
P_1 = \sum_{(i,j) \in R_1} x_{ij}^1 \quad ; \quad P_2 = \sum_{(i,j) \in R_2} x_{ij}^2
\]

A solution \( X_2 \) is said to be better than solution \( X_1 \) either when \( T_2 < T_1 \) or when

\[
T_2 = T_1, \quad \sum_{(i,j) \in R_2} x_{ij}^2 < \sum_{(i,j) \in R_1} x_{ij}^1 \quad \text{i.e.,} \quad P_2 < P_1
\]

(iii) Optimal Solution: A feasible solution for which

\[\text{(Max } t_{ij} : x_{ij} > 0\text{)}\]

is minimal while no better feasible solutions exist is said to be optimal.

**Theorem 1**: If there is a feasible solution to a set of equations \( AX = b, X \geq 0 \), then there is a basic feasible solution.

Hadley [5, p.30] proves this theorem by reducing the number of positive variables in a given feasible solution one by one until the columns of \( A \) associated with positive variables are linearly independent. It may be noted that in
the proof of this theorem, the set of positive variables in
the basic feasible solution is a subset of the positive
variables in the given feasible solutions. The values of
the variables in the two sets, however, may be different.

**Theorem 2:** A locally optimal solution to problem (I) is
also a globally optimal solution.

For the proof of this theorem see Hammer [4] and Swarc
[6].

On the basis of theorems 1 and 2, we develop the
following three-step new algorithm:

**Step 1:** Determine an initial basic feasible solution
by any available method for the cost minimizing
transportation problem.

**Step 2:** Find an adjacent better basic feasible
solution.

**Step 3:** Perform step 2 until no better adjacent basic
feasible solution can be obtained.

Step 2 deals with the determination of a cell (i,j)
not in the basis which if introduced into the basis will
reduce the time of transportation or reduce the allocation
\( x_{hk} \) for at least one \( (h,k) \in R \). This is done in such a way
that the allocation is reduced in that cell belonging to R
in which the allocation is maximum. Hence step 2 consists
of the following substeps:

1. **Find the greatest** \( x_{hk}, (h,k) \in R \).
2. **Identify the set** \( S_{hk} \) **of all cells** (i,j) **not in the
basis, such that if one of these cells is intro-
duced into the basis it will eliminate the cell
(h,k) or reduce the amount** \( x_{hk} \).
3. **Choose among the elements of** \( S_{hk} \) **the one, say,
(\(p,q\)) for which** \( t_{pq} \) **is minimum.
4. **Introduce the cell** (p,q) **into the basis."
The set \( S_{hk} \) is determined as follows: Define a cost matrix.

\[
C_{ij} = \begin{cases} 
1 & \text{if } t_{ij} \geq T \\
0 & \text{if } t_{ij} < T 
\end{cases}
\]

Using \( C = \{C_{ij}\} \) as the cost matrix associated with the initial feasible solution \( X_1 = \{x_{ij}^1\} \), determine \( u_i \) for rows and \( v_j \) for columns by the \((u-v)\) method \(2\) and \( d_{rs} = \{C_{rs} - (u_r + v_s) : (r,s) \notin M\} \). Cells eligible for entry into the basis are those for which \( d_{rs} < 0 \) which implies \( u_r + v_s = 1 \) and \( C_{rs} = 0 \). Thus \( S_{hk} = \{(r,s) : (r,s) \notin M, u_r + v_s = 1, C_{rs} = 0, t_{ij} < T\} \). If \( S_{hk} = \emptyset \), the process terminates. If \( S_{hk} \neq \emptyset \), go to step 2.3.

Note that there is no loss of generality by assuming that the problem is non-degenerate, since degeneracy can be handled in the same manner as in the ordinary transportation problem.

**Numerical Example:** Let there be four producers supplying 37, 22, 32 and 14 units, respectively, with six consumers demanding 15, 20, 15, 25, 20 and 10 units, respectively, with the following transportation time, \( t_{ij} \), \( i = 1,2,\ldots,4 \), \( j = 1,2,\ldots,6 \):

\[
\begin{array}{ccccccc}
& a_i & & & & & \\
& 25 & 30 & 20 & 40 & 45 & 35 & 37 \\
& 30 & 25 & 20 & 30 & 40 & 20 & 22 \\
& 40 & 20 & 40 & 35 & 45 & 22 & 32 \\
& 25 & 24 & 50 & 27 & 30 & 25 & 14 \\
b_j & 15 & 20 & 15 & 25 & 20 & 10 & \\
\end{array}
\]

The North-west Corner rule gives the following initial feasible solution \( X_1 \):
Then

\[ M_1 = \{(i,j) : x_{ij} > 0\} \]
\[ = \{(1,1),(1,2),(1,3),(2,3),(2,4),(3,4),(3,5),(4,5),\]
\[ (4,6)\} \]

\[ T_1 = \max_{(i,j) \in M_1} t_{ij} = 45, \quad R_1 = \{(3,5)\}, \quad \text{hence} \]
\[ (i,j) = (3,5) \]

\[ (h,k) = (3,5) \]

Define Cost Matrix \(C_{ij}\) as follows

\[ C_{ij} = \begin{cases} 
1 & \text{if } t_{ij} > 45 \\
0 & \text{if } t_{ij} \leq 45 
\end{cases} \]

We deduce that \(S_{35}^1 = \{(1,6),(2,5),(2,6),(3,6)\}\). The smallest \(t_{ij}\) with \((i,j) \in S_{35}^1\) being \(t_{26} = 20\), we bring this cell into the basis; this reduces the allocation of \((3,5) = (h,k)\). Thus the new solution \(X_2\) is given by
Proceeding in the same manner we get an optimal solution (with time \( T = 40 \)) given below

\[
\begin{array}{cccccc}
15 & 13 & 9 & & & 0 \\
6 & 6 & 10 & 0 & & \\
7 & 25 & & & 0 & \\
& 14 & & -1 & & \\
0 & 0 & 0 & 0 & 1 & 0
\end{array}
\]

3. The Three-Dimensional Time-Minimizing Transportation Problem

This problem was mathematically defined as PROBLEM II. Again, the objective function is not convex. We use the same definitions and notation as in Section 2 (for details see reference [7]).

**Theorem 3:** A locally optimal solution to PROBLEM II is also globally optimal.

**Proof:** Consider a locally optimal solution \( X_1 = \{x_{ijk}^1\} \) to PROBLEM II. Assume there exists a better solution \( X_2 = \{x_{ijk}^2\} \). Consider a maximization problem defined by constraints (2) and the matrix \( \{C_{ijk}\} \) as follows:

\[
C_{ijk} = \begin{cases} 
1 & \text{if } t_{ijk} < T_1 \\
0 & \text{if } t_{ijk} \geq T_1 
\end{cases}
\]

\[
C_1 = \sum_{(i,j,k) \in M_1} C_{ijk} x_{ijk}^1 = \sum_{(i,j,k) \in (M_1 - R_1)} x_{ijk}^1 = K - p_1
\]
where \( K = \sum_{i=1}^{m} a_i \), \( \sum_{j=1}^{n} b_j = \sum_{k=1}^{p} c_k \)

\[ C_2 = \text{return of solution } X_2 = \sum_{(i,j,k) \in M_2} C_{ijk} x_{ijk}^2 \]

Since \( X_2 \) is a better solution than \( X_1 \), then either \( T_2 < T_1 \) or \( T_2 = T_1, p_2 < p_1 \). When \( T_2 < T_1 \), \( C_{ijk} = 1 \) for occupied cells of \( X_2 \). Therefore

\[ C_2 = \sum_{(i,j,k) \in M_2} C_{ijk} x_{ijk}^2 = \sum_{(i,j,k) \in M_2} x_{ijk}^2 = K. \]

Since \( K > K-p_1 \) it follows that \( C_2 > C_1 \).

When \( T_2 = T_1, p_2 < p_1 \), then

\[ C_2 = \sum_{(i,j,k) \in M_2} C_{ijk} x_{ijk}^2 = \sum_{(i,j,k) \in (M_2-R_2)} x_{ijk}^2 = K - p_2. \]

Since \( K-p_2 > K-p_1 \), it follows that \( C_2 > C_1 \).

Thus, \( X_1 \) cannot be the optimal solution to the maximizing problem. Therefore, there exists an adjacent basic feasible solution \( X_2 \) which is better than \( X_1 \) such that \( C_3 > C_1 \), where \( C_3 \) is the return of solution \( X_3 \).

Hence, \( T_3 < T_1 \).

Let \( T_3 < T_1 \). Now there may exist a possibility in which \( X_3 \) contains all the cells of \( X_1 \) with time \( T_1 \) and the allocations are the same as in \( X_1 \). Then

\[
C_3 = \sum_{(i,j,k) \in M_3} C_{ijk} x_{ijk}^3 = \sum_{(i,j,k) \in (M_3-R_1 \cup R_3)} x_{ijk}^3 = K - p_3 - p_1.
\]

As \( C_1 = K-p_1 \) and \( K-p_1 > K-p_3-p_1 \), it follows that \( C_3 < C_1 \).

This contradicts the fact that \( X_3 \) is better than \( X_1 \).

Therefore \( T_3 \neq T_1 \), hence \( T_3 < T_1 \).
When $T_3 = T_1$,
\[
C_3 = \sum_{(i,j,k) \in M_3} C_{ijk} x_{ijk}^3 = \sum_{(i,j,k) \in (M_3 - R_3)} x_{ijk}^3
\]
\[
= K - p_3.
\]
Hence $C_3 > C_1$ since $K - p_3 > K - p_1$ and $p_3 < p_1$. Thus either $T_3 < T_1$ or $T_3 = T_1$, $p_3 < p_1$.
This implies that $X^3$ is a better basic feasible solution with respect to time, and that $X_1$ is not locally optimal. This contradicts the hypothesis. Hence a locally optimal solution is also globally optimal.

The following solution procedure will converge to the optimal solution to PROBLEM II in a finite number of iterations:

Let $X = \{x_{ijk}\}$ be any basic feasible solution to PROBLEM II yielding time $T$ of transportation which can be obtained in the same way as in (7). Let $M = \{(i,j,k) : x_{ijk} > 0\}$ and $R = \{(i,j,k) : t_{ijk} = T, (i,j,k) \in M\}$. Define a return matrix $[C_{ijk}]$ as in Theorem 3, i.e.
\[
C_{ijk} = \begin{cases} 
1 & \text{if } t_{ijk} < T \\
0 & \text{if } t_{ijk} \geq T 
\end{cases}
\]
For this return maximizing problem, determine dual variables $u_i, v_j, w_k [1]$. Since the return is to be maximized, the cells eligible for entry into the basis are those for which $d_{ghk} > 0$, where
\[
d_{ghk} = (C_{ghk} - (u_g + v_h + w_k) : (g,h,k) \notin M). \quad \text{Note that as } u_g + v_h + w_k = 1 \text{ for } (g,h,k) \in (M-R), \text{ all } v_g, u_h, w_k \text{ are either 0 or 1.}
\]
For cells $(g,h,k)$ with $t_{ghk} < T$, $C_{ghk} = 0$ and therefore $d_{ghk}$ will never be positive. For cells $(g,h,k)$ with $t_{ghk} < T$, $C_{ghk} = 1$, and therefore $d_{ghk}$ is positive only if $u_g + v_h + w_k = 0$. Thus $S_{ghk} = \{(g,h,k) : (g,h,k) \notin M, u_g + v_h + w_k = 0, C_{ghk} = 1\}$. The procedure is repeated until
\[ S_{ghk} = \phi. \]

The procedure is bound to converge for it involves movement from one basic feasible solution to another, which are finite in number. Note again that degeneracy can be handled as explained by Corban [1].

Let us consider the following example with four origins, having \( a_1 = 24, a_2 = 8, a_3 = 18, a_4 = 10 \); four independent destinations, having \( b_1 = 11, b_2 = 19, b_3 = 21, b_4 = 9 \); and three means of transport with \( c_1 = 17, c_2 = 31, c_3 = 12 \), as depicted in Figure 1.

**Figure 1:** Three-dimensional time-minimizing transportation problem.

![Three-dimensional time-minimizing transportation problem](image)

An initial basic feasible solution \( X_1 \) is shown in Figure 2.
Here $T_1 = 15$, $R_1 = \{(2,3,2)\}$. Hence $(g,h,k) = (2,3,2)$, $P_1 = 2$. Define a return matrix as $C_{i,j,k} = \{1 \text{ if } t_{i,j,k} < 15; 0 \text{ if } t_{i,j,k} \geq 15\}$. We deduce that $S_{232} = \{(1,3,1),(1,3,2), (1,3,3),(1,4,1),(1,4,2),(1,4,3),(2,3,3),(2,4,3)\}$. Therefore the cell with minimum time of transportation belonging to $S_{232}$ is $(1,3,2)$, and is introduced into the basis. We obtain a new basic feasible solution $X_2$ shown in Figure 3. Proceeding in the same manner we get an optimal solution (with time $T = 10$), as shown in Figure 4.

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Figure 3: New solution

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Figure 4: Optimal solution