UNIFORM APPROXIMATION OF ENTIRE FUNCTIONS ON COMPACT SETS AND THEIR GENERALIZED GROWTH

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Abstract. In the present paper, we study the polynomial approximation of entire functions of several complex variables. The characterization of generalized order and generalized type of entire function of several complex variables have been obtained in terms of approximation and interpolation errors.

1. Introduction

The concept of generalized order and generalized type for entire transcendental functions was given by Seremeta [4] and Shah [5]. Hence, let \( L^0 \) denote the class of functions \( h \) satisfying the following conditions:

(i) \( h(x) \) is defined on \([a, \infty)\) and is positive, strictly increasing, differentiable and tends to \( \infty \) as \( x \to \infty \),

(ii) \( \lim_{x \to \infty} \frac{h((1+1/\psi(x))x)}{h(x)} = 1 \), for every function \( \psi(x) \) with \( \psi(x) \to \infty \) as \( x \to \infty \).

Let \( \Lambda \) denote the class of functions \( h \) satisfying conditions (i) and

(iii) \( \lim_{x \to \infty} \frac{h(cx)}{h(x)} = 1 \), for every \( c > 0 \), that is \( h(x) \) is slowly increasing.

For an entire transcendental function \( f(z) = \sum_{n=1}^{\infty} b_n z^n \), we put \( M(r) = \max_{|z|=r} |f(z)| \) and functions \( \alpha(x) \in \Lambda, \beta(x) \in L^0 \), then the generalized order of \( f(z) \) is given by

\[ \rho(\alpha, \beta, f) = \lim_{r \to \infty} \sup \frac{\alpha[\log M(r)]}{\beta[\log r]} \]

Further, for \( \alpha(x), \beta^{-1}(x) \) and \( \gamma(x) \in L^0 \), generalized type of an entire transcendental function \( f(z) \) is given as

\[ \sigma(\alpha, \beta, \rho, f) = \lim_{r \to \infty} \sup \frac{\alpha[\log M(r)]}{\beta[\gamma(r)]^{\rho}} \]

where \( 0 < \rho < \infty \) is a fixed number.

Let \( g: C^N \to C \), \( N \geq 1 \), be an entire transcendental function. For \( z = (z_1, z_2, \ldots, z_N) \in C^N \), we put

\[ S(r, g) = \sup \{|g(z)| : |z_1|^2 + |z_2|^2 + \ldots + |z_N|^2 = r^2 \}, \quad r > 0. \]
Then we define the generalized order and generalized type of \( g(z) \) as

\[
\rho(\alpha, \beta, g) = \lim_{r \to \infty} \sup \frac{\alpha \log S(r, g)}{\beta(\log r)}
\]

and

\[
\sigma(\alpha, \beta, \rho, g) = \lim_{r \to \infty} \sup \frac{\alpha \log S(r, g)}{\beta(\log r)} \cdot \frac{\rho}{\gamma(r)}
\]

Let \( K \) be a compact set in \( C^N \) and let \( || \cdot ||_K \) denote the sup norm on \( K \). Given a function \( f \) defined and bounded on \( K \), we put for \( n = 1, 2, \ldots \)

\[
E_1(f, K) = ||f - t_n||_K
\]

\[
E_2(f, K) = ||f - l_n||_K
\]

\[
E_3(f, K) = ||l_{n+1} - l_n||_K
\]

where \( t_n \) denotes the \( n \)-th Chebyshev polynomial of the best approximation to \( f \) on \( K \) and \( l_n \) denotes the \( n \)-th Lagrange interpolation for \( f \) with nodes at extremal points of \( K \) (see [2] and [3]). Let \( u_1, u_2, \ldots, u_n \in K \), where \( u_l = (u_{11}, u_{12}, \ldots, u_{N1}) \).

Following [6], we define

\[
V_n = \max_K \left| \prod_{i=1}^{N} V(u_{i1}, u_{i2}, \ldots, u_{in}) \right|
\]

where \( V(u_{i1}, u_{i2}, \ldots, u_{in}) \) is the Vandermonde determinant for the \( i \)-th co-ordinates of these points, that is,

\[
V(u_n) = \begin{vmatrix} 1 & u_{11} & u_{12}^2 & \cdots & u_{1n}^{n-1} \\ 1 & u_{21} & u_{22}^2 & \cdots & u_{2n}^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & u_{n1} & u_{n2}^2 & \cdots & u_{nn}^{n-1} \end{vmatrix}
\]

Also let \( \mu_n \) denote the smallest maximum modulus of \( n \)-th Chebyshev polynomial \( t_n \) on compact set \( K \). G.M. Goluzin (see [1] p. 296) obtained the relation between \( \mu_n \) and \( \{V_{n+1}/V_n\} \). Here we extend this result for several complex variables. Hence here we write

\[
\prod_{i=1}^{N} [(z_i - u_{i1})(z_i - u_{i2}) \ldots (z_i - u_{in})] = \prod_{i=1}^{N} V(z_i, u_{i1}, u_{i2}, \ldots, u_{in}) / \prod_{i=1}^{N} V(u_{i1}, u_{i2}, \ldots, u_{in}) . \tag{1.1}
\]

Now if the points \( z, u_1, u_2, \ldots, u_n \in K \) and \( \prod_{i=1}^{N} V(u_{i1}, u_{i2}, \ldots, u_{in}) = V_n \) then the modulus of the right hand side of (1.1) does not exceed \( \{V_{n+1}/V_n\} \). Also for \( z \in K \) the modulus of the left hand side is no less than \( \mu_n \). So we get

\[
\mu_n \leq \frac{V_{n+1}}{V_n} \tag{1.2}
\]
Also we have
\[
\prod_{i=1}^{N} V(u_{i1}, u_{i2}, \ldots, u_{i(n+1)}) \leq \prod_{i=1}^{N} \{ |u_{i1}^n| V(u_{i2}, u_{i3}, \ldots, u_{i(n+1)}) | + |u_{i2}^n| V(u_{i1}, u_{i3}, \ldots, u_{i(n+1)}) | + |u_{i(n+1)}^n| V(u_{i1}, u_{i2}, \ldots, u_{in}) \}. \tag{1.3}
\]
Now if the points \( u_1, u_2, \ldots, u_n, u_{n+1} \in K \) and
\[
\prod_{i=1}^{N} V(u_{i1}, u_{i2}, \ldots, u_{in}, u_{i(n+1)}) = V_{n+1}
\]
then from (1.3), we get, successively
\[
V_{n+1} \leq V_n \left[ \prod_{i=1}^{N} \left( |u_{i1}^n| + |u_{i2}^n| + \ldots + |u_{i(n+1)}^n| \right) \right]
\]
\[
\frac{V_{n+1}}{V_n} \leq \prod_{i=1}^{N} |u_{i1}^n| + \prod_{i=1}^{N} |u_{i2}^n| + \ldots + \prod_{i=1}^{N} |u_{i(n+1)}^n|
\]
\[
\frac{V_{n+1}}{V_n} \leq \mu_n (n+1). \tag{1.4}
\]
Finally from (1.2) and (1.4), we get
\[
\mu_n \leq \frac{V_{n+1}}{V_n} \leq \mu_n (n+1).
\]
From the above inequality we get
\[
\lim_{n \to \infty} \left[ \frac{V_{n+2}}{V_{n+1}} \right]^{1/n} = d,
\]
where \( d \) is the transfinite diameter of the compact set \( K \).

Before proving our main results, we state and prove a lemma.

**Lemma 1.** Let \( K \subseteq \mathbb{C}^N \) be a compact set with non-zero transfinite diameter. Let \( f \) be a continuous function on \( K \). Then the function \( f \) can be continuously extended to an entire function \( g(z) \) if and only if
\[
\lim_{n \to \infty} \left[ E_n^s(f, K) \frac{V_{n+1}}{V_{n+2}} \right]^{1/n} = 0; \quad s = 1, 2, 3.
\]

**Proof.** Following [3] and [7], it follows that the function \( f \) can be continuously extended to an entire function \( g(z) \) if and only if
\[
\lim_{n \to \infty} [E_n^s(f, K)]^{1/n} = 0; \quad s = 1, 2, 3.
\]

Also we have
\[
\lim_{n \to \infty} \left[ \frac{V_{n+2}}{V_{n+1}} \right]^{1/n} = d.
\]
Since the transfinite diameter of \( K \) is finite, we get
\[
\lim_{n \to \infty} \left[ E_n^{s}(f, K) \frac{V_{n+1}}{V_{n+2}} \right]^{1/n} = 0.
\]
Hence the lemma is proved. \( \square \)
2. Main Results

Now we prove:

**Theorem 2.1.** Let \( K \subseteq C^N \) be a compact set with non-zero transfinite diameter. If \( \alpha(x) \in A, \beta(x) \in L^0, \) then the function \( f, \) defined and bounded on \( K, \) is the restriction to \( K \) of an entire function \( g \) of generalized order \( \rho(\alpha, \beta, g) \) \((0 < \rho(\alpha, \beta, g) < \infty)\) if and only if

\[
\rho(\alpha, \beta, g) = \lim_{n \to \infty} \sup \frac{\alpha(n)}{\beta \left[ -\frac{1}{n} \log \left( E_n(f, K)^{\frac{V_{n+1}}{V_{n+2}}} \right) \right]} ; \quad s = 1, 2, 3.
\]

**Proof.** Let \( g \) be an entire transcendental function. Write \( \rho = \rho(\alpha, \beta, g) \) and

\[
\theta_s = \lim_{n \to \infty} \sup \frac{\alpha(n)}{\beta \left[ -\frac{1}{n} \log \left( E_n(f, K)^{\frac{V_{n+1}}{V_{n+2}}} \right) \right]} ; \quad s = 1, 2, 3.
\]

Here \( E_n^s \) stands for \( E_n^s(g|K, K), s = 1, 2, 3. \) We claim that \( \rho = \theta_s, s = 1, 2, 3. \) It is known (see e.g. [\( ^7 \)]) that

\[
E_1^1 \leq E_2^2 \leq (n_0 + 2) E_3^3, \quad n \geq 0, \tag{2.1}
\]

\[
E_3^3 \leq 2(n_0 + 2) E_1^1, \quad n \geq 1, \tag{2.2}
\]

where \( n_0 = (c\pi + N). \) Using Stirling formula for the approximate value of

\[
n! \approx e^{-n} n^{n+1/2} \sqrt{2\pi},
\]

we get \( n_0 \approx \frac{2n}{N} \) for all large values of \( n. \) Hence for all large values of \( n, \) we have

\[
E_1^1 \leq E_2^2 \leq \frac{n^N}{N!} (1 + o(1)) E_1^1
\]

and

\[
E_3^3 \leq 2 \frac{n^N}{N!} (1 + o(1)) E_1^1.
\]

Thus \( \theta_3 \leq \theta_2 = \theta_1 \) and it suffices to prove that \( \theta_1 \leq \rho \leq \theta_3. \) First we prove that \( \theta_1 \leq \rho. \) Using the definition of order, for \( \varepsilon > 0 \) and \( r > r_0(\varepsilon), \) we have

\[
S(r, g) \leq \exp \left\{ \alpha^{-1} [\overline{p} \beta(\log r)] \right\},
\]

where \( p = \rho + \varepsilon, \) provided \( r \) is sufficiently large. Without loss of generality, we may suppose that

\[
K \subset B = \{ z \in C^N : |z_1|^2 + |z_2|^2 + \ldots + |z_N|^2 \leq 1 \}.
\]

Then

\[
E_n^1 \frac{V_{n+1}}{V_{n+2}} \leq E_n^1(g, B) \frac{V_{n+1}}{V_{n+2}}.
\]

Now following [\( ^3 \) p. 324], we have

\[
E_n^1(g, B) \frac{V_{n+1}}{V_{n+2}} \leq r^{-n} S(r, g), \quad r \geq 2, \quad n \geq 0
\]

or

\[
E_n^1 \frac{V_{n+1}}{V_{n+2}} \leq r^{-n} \exp \left\{ \alpha^{-1} [\overline{p} \beta(\log r)] \right\}.
\]
Putting \( r = \exp \{ \beta^{-1} \left[ \frac{1}{\beta} \alpha(n) \right] \} \) in the above inequality, we get
\[
E_n^1 \frac{V_{n+1}}{V_{n+2}} \leq \exp \left\{ n - n\beta^{-1} \left[ \frac{1}{\beta} \alpha(n) \right] \right\}
\]
or
\[
\frac{\alpha(n)}{\beta \left( 1 - \frac{1}{n} \log \left( \frac{E_n^1 V_{n+1}}{V_{n+2}} \right) \right)} \leq \overline{\rho}.
\]
Taking limits as \( n \to \infty \), we get
\[
\lim_{n \to \infty} \frac{\alpha(n)}{\beta \left( 1 - \frac{1}{n} \log \left( \frac{E_n^1 V_{n+1}}{V_{n+2}} \right) \right)} \leq \overline{\rho}
\]
or
\[
\theta_1 \leq \overline{\rho}.
\]
Since \( \varepsilon > 0 \) is arbitrarily small, we get
\[
\theta_1 \leq \rho. \tag{2.3}
\]
Now we will prove that \( \rho \leq \theta_3 \). If \( \theta_3 = \infty \), then there is nothing to prove. So let us assume that \( 0 \leq \theta_3 < \infty \). Therefore for all \( \varepsilon > 0 \) there exist \( n_0 \in \mathbb{N} \) such that for all \( n > n_0 \), we have
\[
0 \leq \frac{\alpha(n)}{\beta \left[ -\frac{1}{n} \log \left( E_n^3 (f, K) \frac{V_{n+1}}{V_{n+2}} \right) \right]} \leq \theta_3 + \varepsilon = \overline{\beta}_3
\]
or
\[
E_n^3 \frac{V_{n+1}}{V_{n+2}} \leq \exp \left\{ -n\beta^{-1} \left[ \frac{1}{\beta} \alpha(n) \right] \right\}.
\]
From the property of maximum modulus, we have
\[
S(r, g) \leq \sum_{n=0}^{\infty} E_n^3 \frac{V_{n+1}}{V_{n+2}} r^n
\]
or
\[
S(r, g) \leq \sum_{n=n_0+1}^{n_0} E_n^3 \frac{V_{n+1}}{V_{n+2}} r^n + \sum_{n=n_0+1}^{\infty} r^n \exp \left\{ -n\beta^{-1} \left[ \frac{1}{\beta} \alpha(n) \right] \right\}.
\]
For \( r > 1 \), we have
\[
S(r, g) \leq A_1 r^{n_0} + \sum_{n=n_0+1}^{\infty} r^n \exp \left\{ -n\beta^{-1} \left[ \frac{1}{\beta} \alpha(n) \right] \right\}, \tag{2.4}
\]
where \( A_1 \) is a positive real constant. We take
\[
N(r) = \alpha^{-1} \left\{ \overline{\beta}_3 \beta \left[ \log \{(N + 1)r\} \right] \right\}.
\]
If \( r \) is sufficiently large, then from (2.4) and (2.5), we have
\[
S(r, g) \leq A_1 r^{n_0} + r^{N(r)} \sum_{n_0+1 \leq n \leq N(r)} \exp \left\{ -n\beta^{-1} \left[ \frac{1}{\beta} \alpha(n) \right] \right\} + \sum_{n > N(r)} r^n \exp \left\{ -n\beta^{-1} \left[ \frac{1}{\beta} \alpha(n) \right] \right\}
\]
or

\[ S(r, g) \leq A_1 r^{n_0} + r^{N(r)} \sum_{n=0}^{\infty} \exp \left\{ -n \beta^{-1} \left[ \frac{1}{\theta_3} \alpha(n) \right] \right\} \]

\[ + \sum_{n>N(r)} r^n \exp \left\{ -n \beta^{-1} \left[ \frac{1}{\theta_3} \alpha(n) \right] \right\}. \quad (2.6) \]

Now we have

\[ \lim_{n \to \infty} \sup \left\{ \exp \left\{ -n \beta^{-1} \left[ \frac{1}{\theta_3} \alpha(n) \right] \right\} \right\}^{1/n} = 0. \]

Hence the first series in (2.6) converges to a positive real number \( A_2 \). So from (2.6), we get, successively

\[ S(r, g) \leq A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{n>N(r)} r^n \left[ (N+1) r \right]^{-n} \]

\[ S(r, g) \leq A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{n>N(r)} \left( \frac{1}{N+1} \right)^n \]

\[ S(r, g) \leq A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{n=0}^{\infty} \left( \frac{1}{N+1} \right)^n. \quad (2.7) \]

Now we have

\[ \lim_{n \to \infty} \sup \left\{ \left( \frac{1}{N+1} \right)^n \right\}^{1/n} = \frac{1}{N+1} < 1. \]

Hence the series in (2.7) converges to a positive real constant \( A_3 \). Therefore from (2.7), we get, successively

\[ S(r, g) \leq A_1 r^{n_0} + A_2 r^{N(r)} + A_3 \]

\[ S(r, g) \leq A_2 r^{N(r)} \left[ 1 + o(1) \right] \]

\[ \log S(r, g) \leq [1 + o(1)] N(r) \log r \]

\[ \log S(r, g) \leq [1 + o(1)] \alpha^{-1} \left\{ \beta_3 \beta \left[ \log \{(N + 1) r \} \right] \right\} \log r \]

\[ \log S(r, g) \leq [1 + o(1)] \left[ \alpha^{-1} \left\{ \beta_3 + \delta_1 \beta \left[ \log \{(N + 1) r \} \right] \right\} \right], \]

where \( \delta_1 > 0 \) is suitably small. Hence

\[ \alpha \left[ \log S(r, g) \right] \leq \left( \beta_3 + \delta_1 \right) \beta \left[ \log \{(N + 1) r \} \right], \]

or

\[ \frac{\alpha \left[ \log S(r, g) \right]}{\beta \log r} \leq \left( \beta_3 + \delta_1 \right) \left[ 1 + o(1) \right]. \]

Proceeding to limits as \( r \to \infty \), since \( \delta_1 \) is arbitrarily small, we get

\[ \rho \leq \theta_3. \quad (2.8) \]
Now let $f$ be a function defined and bounded on $K$ and such that for $s = 1, 2, 3$

$$
\theta_s = \lim_{n \to \infty} \sup_{\alpha} \frac{\alpha(n)}{\beta \left[ -\frac{1}{n} \log \left( E_n^s(f, K) V_{n+1}^{V_{n+2}} \right) \right]}
$$

is finite. We claim that the function

$$
g = l_0 + \sum_{n=1}^{\infty} (l_n - l_{n-1})
$$

is the required entire continuation of $f$ and $\rho(\alpha, \beta, g) = \theta_s$. Indeed, for every $\lambda > \theta_s$

$$
\frac{\alpha(n)}{\beta \left[ -\frac{1}{n} \log \left( E_n^s(f, K) V_{n+1}^{V_{n+2}} \right) \right]} \leq \lambda
$$

provided $n$ is sufficiently large. Hence

$$
E_n^s V_{n+1}^{V_{n+2}} \leq \exp \left\{ -n \beta^{-1} \left[ \frac{1}{\lambda} \alpha(n) \right] \right\}.
$$

Using the inequalities (2.1), (2.2) and the proof of converse part given above, we find that the function $g$ is entire and $\rho(\alpha, \beta, g)$ is finite. So by (2.3), we have

$$
\rho(\alpha, \beta, g) = \theta_s,
$$

as claimed. This completes the proof of Theorem 2.

Next we prove:

**Theorem 2.2.** Let $K \subseteq C^N$ be a compact set with non-zero transfinite diameter. If $\alpha(x) \in \Lambda$, $\beta(x) \in L^0$, then the function $f$, defined and bounded on $K$, is the restriction to $K$ of an entire function $g$ of generalized type $\sigma(\alpha, \beta, \rho, g)$ ($0 < \sigma(\alpha, \beta, \rho, g) < \infty$) if and only if

$$
\sigma(\alpha, \beta, \rho, g) = \lim_{n \to \infty} \sup_{\alpha(n/\rho)} \frac{\alpha(n/\rho)}{\beta \left\{ \gamma \left( e^{1/\rho} \left( E_n^s(f, K) V_{n+1}^{V_{n+2}} \right)^{-1/n} \right)^{\rho} \right\}^s} ; \quad s = 1, 2, 3,
$$

where $0 < \rho < \infty$.

**Proof.** Let $g$ be an entire transcendental function. Write $\sigma = \sigma(\alpha, \beta, \rho, g)$ and

$$
\eta_s = \lim_{n \to \infty} \sup_{\alpha(n/\rho)} \frac{\alpha(n/\rho)}{\beta \left\{ \gamma \left( e^{1/\rho} \left( E_n^s(f, K) V_{n+1}^{V_{n+2}} \right)^{-1/n} \right)^{\rho} \right\}^s} ; \quad s = 1, 2, 3.
$$

Here $E_n^s$ stands for $E_n^s(g|K, K)$, $s = 1, 2, 3$. We claim that $\sigma = \eta_s$, $s = 1, 2, 3$. As in the previous theorem, we prove that $\eta_1 \leq \sigma \leq \eta_3$. First we prove that $\eta_1 \leq \sigma$.

Using the definition of the generalized type, for $\varepsilon > 0$ and $r > r_0(\varepsilon)$, we have

$$
S(r, g) \leq \exp \left[ \alpha^{-1} \left( \sigma \beta \left( \gamma(r) \right)^{\rho} \right) \right],
$$

where $\sigma = \sigma + \varepsilon$ provided $r$ is sufficiently large. Again from [3] p. 324, we have

$$
E_n^1 V_{n+1}^{V_{n+2}} \leq r^{-n} S(r, g) \leq r^{-n} \exp \left[ \alpha^{-1} \left( \sigma \beta \left( \gamma(r) \right)^{\rho} \right) \right].
$$
Putting \( r = \gamma^{-1} \left[ \left\{ \beta^{-1} \left( \frac{1}{\eta_3} \alpha(n/\rho) \right) \right\}^{1/\rho} \right] \) in the above inequality, we get
\[
E_n^1 V_{n+1} V_{n+2} \leq \exp(n/\rho) \left[ \gamma^{-1} \left\{ \beta^{-1} \left( \frac{1}{\eta_3} \alpha(n/\rho) \right) \right\}^{1/\rho} \right]^{-n}
\]
or
\[
\frac{\alpha(n/\rho)}{\beta \left\{ \gamma \left( \frac{1}{\rho} \left\{ E_n^1 (f, K) \frac{V_{n+1}}{V_{n+2}} \right\}^{-1/n} \right) \right\}^\rho} \leq \sigma = \sigma + \varepsilon.
\]
Letting \( n \to \infty \) and proceeding to limits, we get
\[
\lim_{n \to \infty} \sup \frac{\alpha(n/\rho)}{\beta \left\{ \gamma \left( \frac{1}{\rho} \left\{ E_n^1 (f, K) \frac{V_{n+1}}{V_{n+2}} \right\}^{-1/n} \right) \right\}^\rho} \leq \sigma
\]
or
\[
\eta_1 \leq \sigma.
\]
Since \( \varepsilon > 0 \) is arbitrary, we have
\[
\eta_1 \leq \sigma. \quad (2.9)
\]
Now we will prove that \( \sigma \leq \eta_3 \). If \( \eta_1 = \infty \), then there is nothing to prove. So let us assume that \( 0 \leq \eta_3 < \infty \). Therefore for all \( \varepsilon > 0 \) there exist \( n_0 \in N \) such that for all \( n > n_0 \), we have
\[
0 \leq \frac{\alpha(n/\rho)}{\beta \left\{ \gamma \left( \frac{1}{\rho} \left\{ E_n^1 (f, K) \frac{V_{n+1}}{V_{n+2}} \right\}^{-1/n} \right) \right\}^\rho} \leq \eta_3 + \varepsilon = \eta_3
\]
or
\[
E_n^3 \frac{V_{n+1}}{V_{n+2}} \leq e^{n/\rho} \left[ \gamma^{-1} \left\{ \beta^{-1} \left( \frac{1}{\eta_3} \alpha(n/\rho) \right) \right\}^{1/\rho} \right]^{-n}.
\]
From the maximum modulus property, we have
\[
S(r, g) \leq \sum_{n=0}^\infty E_n^3 \frac{V_{n+1}}{V_{n+2}} r^n
\]
or
\[
S(r, g) \leq \sum_{n=0}^{n_0} E_n^3 \frac{V_{n+1}}{V_{n+2}} r^n + \sum_{n=n_0+1}^\infty r^n e^{n/\rho} \left[ \gamma^{-1} \left\{ \beta^{-1} \left( \frac{1}{\eta_3} \alpha(n/\rho) \right) \right\}^{1/\rho} \right]^{-n}.
\]
For \( r > 1 \), we have
\[
S(r, g) \leq B_1 r^{n_0} + \sum_{n=n_0+1}^\infty r^n e^{n/\rho} \left[ \gamma^{-1} \left\{ \beta^{-1} \left( \frac{1}{\eta_3} \alpha(n/\rho) \right) \right\}^{1/\rho} \right]^{-n}, \quad (2.10)
\]
where \( B_1 \) is a positive real constant. We take
\[
N(r) = \rho \alpha^{-1} \left[ \eta_3 \beta \left( \left\{ (N + 1) re^{1/\rho} \right\}^\rho \right) \right], \quad (2.11)
\]
If \( r \) is sufficiently large, then from (2.10) and (2.11), we have

\[
S(r, g) \leq B_1 r^{n_0} + r N(r) \sum_{n=n_0+1}^{\infty} e^{n/\rho} \left[ \gamma^{-1} \left\{ \beta^{-1} \left( \frac{1}{\eta_3} \alpha(n/\rho) \right) \right\}^{1/\rho} \right]^{-n} \\
+ \sum_{n>N(r)} e^{n/\rho} \left[ \gamma^{-1} \left\{ \beta^{-1} \left( \frac{1}{\eta_3} \alpha(n/\rho) \right) \right\}^{1/\rho} \right]^{-n} r^n
\]

or

\[
S(r, g) \leq B_1 r^{n_0} + r N(r) \sum_{n=0}^{\infty} e^{n/\rho} \left[ \gamma^{-1} \left\{ \beta^{-1} \left( \frac{1}{\eta_3} \alpha(n/\rho) \right) \right\}^{1/\rho} \right]^{-n} \\
+ \sum_{n>N(r)} e^{n/\rho} \left[ \gamma^{-1} \left\{ \beta^{-1} \left( \frac{1}{\eta_3} \alpha(n/\rho) \right) \right\}^{1/\rho} \right]^{-n} r^n. \tag{2.12}
\]

Now we have

\[
\lim_{n \to \infty} \sup \left( e^{n/\rho} \left[ \gamma^{-1} \left\{ \beta^{-1} \left( \frac{1}{\eta_3} \alpha(n/\rho) \right) \right\}^{1/\rho} \right]^{-n} \right) = 0.
\]

Hence the first series in (2.12) converges to a positive real constant \( B_2 \) and we get successively

\[
S(r, g) \leq B_1 r^{n_0} + B_2 r N(r) + \sum_{n>n_0}^{\infty} e^{n/\rho} \left[ \gamma^{-1} \left\{ \beta^{-1} \left( \frac{1}{\eta_3} \alpha(n/\rho) \right) \right\}^{1/\rho} \right]^{-n} r^n \\
S(r, g) \leq B_1 r^{n_0} + B_2 r N(r) + \sum_{n>n_0}^{\infty} r^n e^{n/\rho} [(N+1)re^{1/\rho}]^{-n} \\
S(r, g) \leq B_1 r^{n_0} + B_2 r N(r) + \sum_{n>n_0}^{\infty} \left( \frac{1}{N+1} \right)^n \\
S(r, g) \leq B_1 r^{n_0} + B_2 r N(r) + \sum_{n=0}^{\infty} \left( \frac{1}{N+1} \right)^n. \tag{2.13}
\]

Now by previous theorem we can say that the series in (2.13) converges to a positive real number \( B_3 \). Hence, successively

\[
S(r, g) \leq B_1 r^{n_0} + B_2 r N(r) + B_3 \\
\log S(r, g) \leq [1 + o(1)] N(r) \log r \\
\log S(r, g) \leq [1 + o(1)] \left( \frac{\rho \alpha^{-1} \left[ \eta_3 \beta \left( \gamma \{ \sigma + 1 \} r \right) \right]^{1/\rho} \right) \log r \\
\log S(r, g) \leq [1 + o(1)] \alpha^{-1} \left[ \frac{1}{\eta_3 + \delta_2} \beta \left( \gamma \{ \sigma + 1 \} r \right) \right]^{1/\rho},
\]

where \( \delta_2 > 0 \) is suitably small. Hence, successively

\[
o [\log S(r, g)] \leq (\eta_3 + \delta_2) \beta \left( \gamma \{ \sigma + 1 \} r \right)^{1/\rho}
\]
\[ \alpha \log S(r, g) \leq \eta_3 + \delta_2 [\eta_3 + \delta_2] \]

\[ \alpha \log S(r, g) \leq \eta_3 + \delta_2 [\eta_3 + \delta_2] [1 + o(1)] \]

\[ \frac{\alpha \log S(r, g)}{\beta ([\gamma(r)]^p)} \leq \eta_3 + \delta_2 [1 + o(1)]. \]

Proceeding to limits as \( r \to \infty \), since \( \delta_2 \) is arbitrarily small, we get

\[ \sigma \leq \eta_3. \quad (2.14) \]

Now let \( f \) be a function defined and bounded on \( K \) and such that for \( s = 1, 2, 3 \)

\[ \eta_s = \lim_{n \to \infty} \sup \frac{\alpha(n/\rho)}{\beta \left\{ \gamma \left( e^{1/\rho} \left[ E^n(f, K) \frac{V_{n+1}}{V_{n+2}} \right]^{-1/n} \right)^{1/p} \right\} \}

is finite. We claim that the function

\[ g = l_0 + \sum_{n=1}^{\infty} (l_n - l_{n-1}) \]

is the required entire continuation of \( f \) and \( \sigma(\alpha, \beta, \rho, g) = \eta_s \). Indeed, for every \( \mu > \eta_s \)

\[ \frac{\alpha(n/\rho)}{\beta \left\{ \gamma \left( e^{1/\rho} \left[ E^n(f, K) \frac{V_{n+1}}{V_{n+2}} \right]^{-1/n} \right)^{1/p} \right\} \} \leq \mu \]

provided \( n \) is sufficiently large. Hence

\[ E^n_{\rho} \frac{V_{n+1}}{V_{n+2}} \leq e^{n/\rho} \left[ \gamma^{-1} \left\{ \beta^{-1} \left( \frac{1}{\mu} \alpha(n/\rho) \right) \right\}^{1/\rho} \right]^{-n}. \]

Using the inequalities (2.1), (2.2) and proof of the converse part given above, we find that the function \( g \) is entire and \( \sigma(\alpha, \beta, \rho, g) \) is finite. So by (2.9), we have \( \sigma(\alpha, \beta, \rho, g) = \eta_s \), as claimed. This completes the proof of Theorem 2.2. \( \Box \)

References