EULER SUMS ON ARITHMETIC PROGRESSIONS

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Abstract. We decompose the classical Euler sum into a linear combination of sums of the form

\[ H_{p,q}(a;c) = \sum_{k=0}^{\infty} \frac{1}{(ak+c)^q} \sum_{j=0}^{\tilde{k}} \frac{1}{(aj+b)^p} \quad (\tilde{k} = k \text{ if } b \leq c, \tilde{k} = k-1 \text{ if } b > c) \]

which we call Euler sums on arithmetic progressions. Through basic linear relations among these new Euler sums, we are able to evaluate the family \( D_{p,q}(a) := \sum_{b=1}^{a} H_{p,q}(a;b,b) \) when the weight \( p + q \) is odd. In addition, we obtain the evaluation of \( T_{p,q}(n) \) no matter when the weight is even or odd and construct a lot of new families which can be evaluated when the weight is odd.

1. Introduction

The classical Euler sums defined by

\[ S_{p,q} := \sum_{k=1}^{\infty} \frac{1}{k^q} \sum_{j=1}^{k} \frac{1}{j^p}, \quad q \geq 2, \quad (1) \]

are investigated [1, 2, 3, 8] and generalized [4, 6, 10] to more general sums defined by

\[ E^{(n)}_{p,q} := \sum_{k=1}^{\infty} \frac{1}{k^q} \sum_{j=1}^{[kn]} \frac{1}{j^p} \quad (2) \]

\[ T^{(n)}_{p,q} := \sum_{k=1}^{\infty} \frac{1}{k^q} \sum_{j=1}^{[k/n]} \frac{1}{j^p}, \quad (3) \]

where \([x]\) is the greatest integer less than or equal to \(x\).

In this paper, we shall decompose the classical and the aforementioned generalized Euler sums into more basic sums, which we call Euler sums on arithmetic progressions. For \(p, q, a, b, c \in \mathbb{Z}^+\) with \(q \geq 2\) and \(b \leq a, c \leq a\), let

\[ H_{p,q}(a;b,c) := H_{p,q}(a;c) = \sum_{k=0}^{\infty} \frac{1}{(ak+c)^q} \sum_{j=0}^{\tilde{k}} \frac{1}{(aj+b)^p}, \quad (4) \]

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where
\[ \tilde{k} = \begin{cases} k, & \text{if } b \leq c; \\ k - 1, & \text{if } b > c. \end{cases} \]

Here and throughout we prefer to keep the parameters \( b/a \) and \( c/a \) of \( H_{p,q} \) as un-reduced fractions. The following identities are immediate from the definitions.

\[
H_{p,q} \left( \frac{b}{a}, \frac{c}{a} \right) = a^{-(p+q)} S_{p,q},
\]

(5)

\[
E_{p,q}^{(a)} = a^p \sum_{b=1}^{a} H_{p,q} \left( \frac{b}{a}, \frac{c}{a} \right),
\]

(6)

\[
T_{p,q}^{(a)} = a^p \sum_{b=1}^{a} H_{p,q} \left( \frac{c}{a}, \frac{b}{a} \right),
\]

(7)

and

\[
S_{p,q} = \sum_{b=1}^{a} \sum_{c=1}^{a} H_{p,q} \left( \frac{b}{a}, \frac{c}{a} \right).
\]

(8)

Also we have the following reflection formulæ: For \( p, q \in \mathbb{Z}^+ \) with \( p, q \geq 2 \),

\[
H_{p,q} \left( \frac{b}{a}, \frac{c}{a} \right) + H_{q,p} \left( \frac{c}{a}, \frac{b}{a} \right) = a^{-(p+q)} \zeta \left( p, \frac{b}{a} \right) \zeta \left( q, \frac{c}{a} \right)
\]

(9)

when \( b \neq c \) and

\[
H_{p,q} \left( \frac{b}{a}, \frac{b}{a} \right) + H_{q,p} \left( \frac{b}{a}, \frac{b}{a} \right) = a^{-(p+q)} \left\{ \zeta \left( p, \frac{b}{a} \right) \zeta \left( q, \frac{b}{a} \right) + \zeta \left( p+q, \frac{b}{a} \right) \right\}. \]

(10)

Not all of \( H_{p,q}(b/a, c/a) \) can be evaluated individually in terms of Hurwitz zeta values (i.e. values at positive integers of Hurwitz zeta functions). However, we can form a suitable finite sum of \( H_{p,q}(b/a, c/a) \) which evaluates nicely. In particular, we consider a family of sums defined by

\[
D_{p,q}^{(a)} := \sum_{b=1}^{a} H_{p,q} \left( \frac{b}{a}, \frac{b}{a} \right)
\]

(11)

and obtain the evaluation of \( D_{p,q}^{(a)} \) when \( p + q \) is odd in Section 3.

In Section 4, we construct other families of various combinations of \( H_{p,q}(b/a, c/a) \) and derive important relations among them. These kinds of relations are also enjoyed among \( S_{p,q} \) of the same weight (see (32) below) when the weight \( p + q \) is odd, and they were used to give explicit evaluation of \( S_{p,q} \) by solving linear systems [2]. Through careful study of these relations, in Section 5 we show that the two families \( K_{p,q}(a,b) \) and \( F_{p,q}(k; a, b) \) can be evaluated in terms of Hurwitz zeta values when the weight is odd.

2. Some Basic Relations and the Evaluation of \( T_{1,n}^{(a)} \)

As usual, let \( \zeta(s) \) and \( \zeta(s, x) \) denote the Riemann and Hurwitz zeta function, respectively, defined for \( \operatorname{Re}(s) > 1 \) and \( x > 0 \) as

\[
\zeta(s, x) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \quad \operatorname{Re} s > 1, \ x > 0.
\]
Note that $\zeta(s) = \zeta(s, 1)$. The Kronecker limit formula for the Hurwitz zeta function is given by (see e.g. [9, p. 91])

$$\lim_{s \to 1^+} \left\{ \zeta(s, x) - \frac{1}{s - 1} \right\} = -\psi(x),$$

where $\psi(x)$ is the familiar logarithmic derivative of the gamma function:

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}.$$

We shall need various identities pertaining to $\psi(x)$.

$$\sum_{j=0}^{\infty} \left\{ \frac{1}{aj + b} - \frac{1}{aj + c} \right\} = \frac{1}{a} \left\{ -\psi\left(\frac{b}{a}\right) + \psi\left(\frac{c}{a}\right) \right\}, \quad \text{for} \quad a, b, c > 0,$$

$$-\psi(1 - x) = \gamma + \sum_{\ell=1}^{\infty} \zeta(\ell + 1)x^{\ell}, \quad \text{for} \quad |x| < 1,$$

$$\psi(x + 1) = \frac{1}{x} + \psi(x), \quad \text{for} \quad x > 0,$$

and

$$\psi(x) - \psi(1 - x) = -\pi \cot \pi x, \quad \text{for} \quad x > 0.$$

Eq. (14) follows easily from (12). Eq. (15) is well known (see e.g. [1, p. 141, Eq. (6.2)]). Eq. (16) and (17) are easy consequences of the functional equation and the logarithmic differentiation of the reflection formula for the gamma function, respectively.

We now proceed to develop relations among $H_{1,n}(b/a, c/a)$ for various $b$ and $c$.

**Proposition 1.** For $a, b, n \in \mathbb{Z}^+$ with $b < a$ and $n \geq 2$, we have

$$a^{n+1}H_{1,n}\left(\frac{a}{a}, \frac{b}{a}\right) = \frac{n}{2}\left( n + 1, \frac{b}{a} \right) + \zeta\left( n + 1, \frac{b}{a} \right) \left\{ \psi\left(\frac{b}{a}\right) + \gamma \right\}$$

$$-\frac{1}{2} \sum_{\ell=2}^{n-1} \zeta\left(\ell, \frac{b}{a}\right) \zeta\left( n + 1 - \ell, \frac{b}{a}\right).$$

**Proof.** By definition, we have

$$a^{n+1}H_{1,n}\left(\frac{a}{a}, \frac{b}{a}\right) = a^{n+1} \sum_{k=0}^{\infty} \frac{1}{(ak + b)^n} \sum_{j=1}^{k} \frac{1}{aj}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(k + r)^n} \sum_{j=1}^{k} \frac{1}{j},$$

where $r = b/a$. Rewrite the above series as

$$\sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \frac{1}{(k + j + r)^n j}.$$

With the help of the partial fraction decomposition

$$\frac{1}{(x + \alpha)^n x} = \frac{1}{\alpha^n} \left( \frac{1}{x} - \frac{1}{x + \alpha} \right) - \sum_{\ell=1}^{n-1} \frac{1}{\alpha^{\ell}(x + \alpha)^{n-\ell}},$$

(18)
we are able to rewrite $a^{n+1} H_{1,n}(a/a, b/a)$ as

$$\sum_{k=0}^{\infty} \frac{1}{(k+r)^n} \sum_{j=1}^{\infty} \left( \frac{1}{j} - \frac{1}{k+j+r} \right) - \sum_{\ell=1}^{n-1} \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \frac{1}{(k+r)^{\ell}(k+j+r)^{n+1-\ell}}.$$  

In view of (15) and (14), the first term in the above is equal to

$$\sum_{k=0}^{\infty} \frac{1}{(k+r)^n} \left\{ \psi(k+1+r) + \gamma \right\},$$

or more explicitly as

$$a^{n+1} H_{1,n}(b/a, b/a) + \zeta(n, b/a) \left\{ \psi(b/a) + \gamma \right\},$$

(19)

by repeatedly using the functional equation (16). The second term is equal to

$$-\sum_{\ell=1}^{n-1} \left\{ \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{(k+r)^{\ell}(k+j+r)^{n+1-\ell}} - \zeta(n+1, r) \right\}$$

$$= -a^{n+1} H_{1,n} \left( \frac{b}{a}, \frac{b}{a} \right) - a^{n+1} \sum_{\ell=2}^{n-1} H_{\ell,n+1-\ell} \left( \frac{b}{a}, \frac{b}{a} \right) + (n-1) \zeta \left( n+1, \frac{b}{a} \right).$$

With the reflection formula (6), the above is equal to

$$\frac{n}{2} \zeta \left( n+1, \frac{b}{a} \right) - a^{n+1} H_{1,n} \left( \frac{b}{a}, \frac{b}{a} \right) - \frac{1}{2} \sum_{\ell=2}^{n-1} \zeta \left( \ell, \frac{b}{a} \right) \zeta \left( n+1-\ell, \frac{b}{a} \right).$$

(20)

assertion then follows from the addition of (19) and (20).

**Remark.** When $a = b$, we have a slightly different formula for

$$a^{n+1} H_{1,n} \left( \frac{a}{a}, \frac{a}{a} \right) = S_{1,n},$$

which recovers the well-known formula of Euler:

$$S_{1,n} = \frac{n+2}{2} \zeta(n+1) - \frac{1}{2} \sum_{\ell=2}^{n-1} \zeta(\ell) \zeta(n+1-\ell).$$

(21)

Now we have our evaluation of $T_{1,n}(a)$.

**Theorem 1.** For $n, a \in \mathbb{Z}^+$ with $n \geq 2$, we have

$$T_{1,n}(a) = \left( \frac{na}{2} + \frac{1}{a^n} \right) \zeta(n+1) + a^{-n} \sum_{b=1}^{a-1} \zeta(n, \frac{b}{a}) \left\{ \psi \left( \frac{b}{a} \right) + \gamma \right\}$$

$$- \frac{1}{2a^n} \sum_{\ell=2}^{n-1} \zeta(\ell, \frac{b}{a}) \zeta(n+1-\ell, \frac{b}{a}).$$

(22)

**Proof.** This follows from (7), Proposition 1, and (21).
Proposition 2. For $a, b, c, n \in \mathbb{Z}^+$ with $b < c < a$ and $n \geq 2$, we have
\[
a^{n+1} \left\{ H_{1,n} \left( \frac{b}{a}, \frac{c}{a} \right) + H_{1,n} \left( \frac{a-b}{a}, \frac{c-b}{a} \right) \right\}
\]
\[= \zeta \left( n, \frac{c-b}{a} \right) \left\{ -\psi \left( \frac{b}{a} \right) + \psi \left( \frac{c}{a} \right) \right\} + \zeta \left( n, \frac{c}{a} \right) \left\{ -\psi \left( 1 - \frac{b}{a} \right) + \psi \left( \frac{c-b}{a} \right) \right\}
\]
\[- \sum_{\ell=2}^{n-1} \zeta(\ell, \frac{c}{a}) \zeta(n+1-\ell, \frac{c-b}{a}).
\]
(23)

Proof. By definition, we have
\[
H_{1,n} \left( \frac{b}{a}, \frac{c}{a} \right) = \sum_{k=0}^{\infty} \frac{1}{(ak+c)^n} \sum_{j=0}^{k} \frac{1}{a_j + b},
\]
which is equal to the double series
\[
\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{(ak+c+a)^n(a_j + b)}.
\]
In light of the partial fraction decomposition (18), we get
\[
H_{1,n} \left( \frac{b}{a}, \frac{c}{a} \right) = \sum_{k=0}^{\infty} \frac{1}{(ak+c-b)^n} \sum_{j=0}^{\infty} \left\{ \frac{1}{a_j + b} - \frac{1}{ak+a_j+c} \right\}
\]
\[- \sum_{\ell=1}^{n-1} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{(ak+c-b)^\ell(ak+a_j+c)^{n+1-\ell}},
\]
or equivalently,
\[
a^{-(n+1)} \zeta \left( n, \frac{c-b}{a} \right) \left\{ -\psi \left( \frac{b}{a} \right) + \psi \left( \frac{c}{a} \right) \right\}
\]
\[+ H_{1,n} \left( \frac{c}{a}, \frac{c-b}{a} \right) - \sum_{\ell=1}^{n-1} H_{\ell,n+1-\ell} \left( \frac{c-b}{a}, \frac{c}{a} \right).
\]
In exactly the same way, we get
\[
H_{1,n} \left( \frac{a-b}{a}, \frac{c-b}{a} \right) = a^{-(n+1)} \zeta \left( n, \frac{c}{a} \right) \left\{ -\psi \left( 1 - \frac{b}{a} \right) + \psi \left( \frac{c-b}{a} \right) \right\}
\]
\[+ H_{1,n} \left( \frac{c-b}{a}, \frac{c}{a} \right) - \sum_{\ell=1}^{n-1} H_{\ell,n+1-\ell} \left( \frac{c}{a}, \frac{c-b}{a} \right).
\]
On using the reflection formula (9) in the following form: for $p, q \geq 2$,
\[
H_{p,q} \left( \frac{c}{a}, \frac{c-b}{a} \right) + H_{q,p} \left( \frac{c-b}{a}, \frac{c}{a} \right) = a^{-(p+q)} \zeta \left( p, \frac{c}{a} \right) \zeta \left( q, \frac{c-b}{a} \right),
\]
we finish the proof.
Proposition 3. For \(a, b, n \in \mathbb{Z}^+\) with \(b < a\) and \(n \geq 2\), we have

\[a^{n+1} \left\{ H_{1,n} \left( \frac{b}{a}, \frac{b}{a} \right) + H_{1,n} \left( \frac{a-b}{a}, \frac{a}{a} \right) \right\} = \zeta(n+1, \frac{b}{a}) - \zeta(\frac{n}{a}) \left\{ \psi \left( 1 - \frac{b}{a} \right) + \gamma \right\} - \sum_{\ell=2}^{n-1} \zeta(\ell) \zeta \left( n+1 - \ell, \frac{b}{a} \right) \cdot \tag{24}\]

Proof. Here we rewrite \(H_{1,n} \left( \frac{b}{a}, \frac{b}{a} \right)\) as

\[\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{(ak + aj + b)^n(aj + b)},\]

or

\[a^{-(n+1)} \zeta(n+1, \frac{b}{a}) + \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \frac{1}{(ak + aj + b)^n(aj + b)}.\]

The rest of calculation is the same as in previous propositions, so we omit it. \(\square\)

3. The Evaluations of \(D_{1,2n}^{(a)}\) and \(D_{2m,2n+1}^{(a)}\)

Recall that

\[D_{p,q}^{(a)} = \sum_{b=1}^{a} H_{p,q} \left( \frac{b}{a}, \frac{b}{a} \right).\]

In order to obtain the evaluation of \(D_{1,2n}^{(a)}\), we need the evaluation of \(E_{1,2n}^{(a)}\).

Theorem A. \[4\] For each positive integer \(n \geq 2\), we have

\[E_{1,2n}^{(a)} = \frac{2n+1}{2a} \zeta(2n+1) - \sum_{\ell=0}^{n-1} a^{2n-2\ell} \zeta(2\ell) \zeta(2n-2\ell+1) - a^{2n-1} \sum_{b=1}^{a-1} S_1 \left( \frac{b}{a} \right) S_{2n} \left( \frac{b}{a} \right), \tag{25}\]

where \(\zeta(0) = -1/2\) and

\[S_\ell(x) = \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n^\ell}.\]

Theorem 2. For positive integers \(a\) and \(n\), we have

\[D_{1,2n}^{(a)} = \left\{ \frac{1}{2} + \frac{1}{2a^{2n+1}} + \frac{n}{a^{2n+1}} \right\} \zeta(2n+1) - \sum_{b=1}^{a-1} a^{-(2n+1)} \zeta \left( 2n, \frac{b}{a} \right) \left\{ \psi \left( 1 - \frac{b}{a} \right) + \gamma \right\} + \frac{1}{a} \sum_{b=1}^{a-1} S_1 \left( \frac{b}{a} \right) S_{2n} \left( \frac{b}{a} \right) - \sum_{\ell=1}^{n-1} a^{-(2\ell+1)} \zeta(2\ell+1) \zeta(2n-2\ell). \tag{26}\]
Proof. By Proposition 3, we have

\[ a^{2n+1} \left\{ \sum_{b=1}^{a} H_{1,2n} \left( \frac{b}{a}, \frac{b}{a} \right) + \sum_{b=1}^{a} H_{1,2n} \left( \frac{b}{a}, \frac{a}{a} \right) \right\} \]

\[ = 2S_{1,2n} + \sum_{b=1}^{a-1} \zeta(2n+1, \frac{b}{a}) - \sum_{b=1}^{a-1} \zeta \left( 2n, \frac{b}{a} \right) \left\{ \psi \left( 1 - \frac{b}{a} \right) + \gamma \right\} \]

\[ - \sum_{\ell=2}^{2n-1} \sum_{b=1}^{a-1} \zeta(\ell) \zeta \left( 2n + 1 - \ell, \frac{b}{a} \right). \]

Our assertion then follows from (6) and the evaluation of \( E^{(a)}_{1,2n} \) given in Theorem A. □

The evaluation of \( D^{(a)}_{2m,2n+1} \) depends on \( E^{(a)}_{1,2m+2n} \) and \( T^{(a)}_{p,q} \) with \( p + q = 2m + 2n + 1 \), so we need the following theorem.

**Theorem B.** \([4]\) For odd weight \( w = p + q \) with \( p > 1 \) and \( q > 1 \), we have

\[ T^{(a)}_{p,q} = \frac{1}{2a^q} \zeta(w) + \frac{1 - (-1)^p}{2} \zeta(p) \zeta(q) \]

\[ + (-1)^p \sum_{\ell=0}^{\lfloor p/2 \rfloor} \left( \frac{w - 2\ell - 1}{p - 1} \right) a^{p-2\ell} \zeta(2\ell) \zeta(w - 2\ell) \]

\[ + (-1)^p a^{p-1} \sum_{\ell=0}^{\lfloor q/2 \rfloor} \left( \frac{w - 2\ell - 1}{p - 1} \right) \zeta(2\ell) \zeta(w - 2\ell) \]

\[ + (-1)^p a^{p-1} \sum_{b=1}^{a-1} \left( \frac{w - 2\ell - 1}{p - 1} \right) C_{2\ell} \left( \frac{b}{a} \right) C_{w-2\ell} \left( \frac{b}{a} \right) \]

\[ + (-1)^p a^{p-1} \sum_{b=1}^{a-1} \left( \frac{w - 2\ell - 2}{p - 1} \right) S_{2\ell+1} \left( \frac{b}{a} \right) S_{w-2\ell-1} \left( \frac{b}{a} \right), \quad (27) \]

where

\[ C_t(x) = \sum_{n=1}^{\infty} \frac{\cos 2n\pi x}{n^t} \]

and

\[ S_t(x) = \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n^t}. \]
Theorem 3. For positive integers \(m\) and \(n\) with \(w = 2m + 2n + 1\), we have
\[
D_{2m,2n+1}^{(a)} = \left\{ a^{-w} \left( \frac{w-2}{2m-1} \right) + 1 \right\} \zeta(w)
\]
\[
+ \sum_{\ell=2}^{2m} \left( \frac{w-\ell-1}{2n} \right) (-1)^\ell a^{-w+\ell} \zeta(w-\ell) \zeta(\ell)
\]
\[
- \left( \frac{w-2}{2m-1} \right) a^{-1} E_{1,2m+2n}^{(a)} + \sum_{\ell=2}^{2n+1} \left( \frac{w-\ell-1}{2m-1} \right) a^{-w-\ell} T_{w-\ell,\ell}^{(a)}.
\]

Proof. First we rewrite \(H_{2m,2n+1}(b/a,b/a)\) as
\[
\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{(ak+aj+b)^{2n+1}(a+b)^{2m}},
\]
which is equal to
\[
a^{-w} \zeta(w, b/a) + \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \frac{1}{(ak+aj+b)^{2n+1}(a+b)^{2m}}
\]
with a change of variable in the summation. To obtain the sum of the above double series, we need the partial fraction decomposition of the rational function
\[
\frac{1}{(x+\alpha)^{2n+1}x^{2m}}.
\]
By differentiating the identity
\[
\frac{1}{(x+\alpha)x^{2m}} = \sum_{\ell=1}^{2m} \frac{(-1)^\ell}{\alpha^{2m+1-\ell}x^\ell} + \frac{1}{\alpha^{2m}(x+\alpha)}
\]
\(2n\) times with respect to \(\alpha\), we get the identity
\[
\frac{1}{(x+\alpha)^{2n+1}x^{2m}} = \sum_{\ell=1}^{2m} \left( \frac{w-\ell-1}{2n} \right) (-1)^\ell \frac{1}{\alpha^{w-\ell}x^\ell}
\]
\[
+ \sum_{\ell=2}^{2n+1} \left( \frac{w-\ell-1}{2m-1} \right) \frac{1}{\alpha^{w-\ell}(x+\alpha)^\ell},
\]
which we rewrite as
\[
\frac{1}{(x+\alpha)^{2n+1}x^{2m}} = \sum_{\ell=2}^{2m} \left( \frac{w-\ell-1}{2n} \right) (-1)^\ell \frac{1}{\alpha^{w-\ell}x^\ell} - \left( \frac{w-2}{2n} \right) \frac{1}{\alpha^{2m+2n}} \left\{ \frac{1}{x} - \frac{1}{x+\alpha} \right\}
\]
\[
+ \sum_{\ell=2}^{2n+1} \left( \frac{w-\ell-1}{2m-1} \right) \frac{1}{\alpha^{w-\ell}(x+\alpha)^\ell}.
\]
Setting $\alpha = ak$ and $x = aj + b$ in (30) and summing together where $k$ ranges over all positive integers and $j$ ranges over all non-negative integers, we get

\[
H_{2m,2n+1} \left( \frac{b}{a}, \frac{b}{a} \right) = a^{-w} \zeta \left( \frac{w}{a}, \frac{b}{a} \right) + a^{-w} \sum_{\ell=2}^{2m} \left( \frac{w - \ell - 1}{2n} \right) (-1)^{\ell} \zeta \left( \frac{w - \ell - 1}{2n} \right) \zeta \left( \frac{\ell}{a}, \frac{b}{a} \right)
\]

\[
- \left( \frac{w - 2}{2n} \right) H_{1,2m+2n} \left( \frac{b}{a}, \frac{a}{a} \right) + \sum_{\ell=2}^{2n+1} \left( \frac{w - \ell - 1}{2m - 1} \right) H_{w - \ell, \ell} \left( \frac{a}{a}, \frac{b}{a} \right). \tag{31}
\]

On the other hand, we set $x = j$, $\alpha = k$ and sum together with $j$ and $k$ ranging over all positive integers to get

\[
S_{2m,2n+1} = \zeta(w) + \sum_{\ell=2}^{2m} \left( \frac{w - \ell - 1}{2n} \right) (-1)^{\ell} \zeta \left( \frac{w - \ell - 1}{2n} \right) \zeta(\ell)
\]

\[
- \left( \frac{w - 2}{2n} \right) S_{1,2m+2n} + \sum_{\ell=2}^{2n+1} \left( \frac{w - \ell - 1}{2m - 1} \right) \{S_{w - \ell, \ell} - \zeta(w)\}, \tag{32}
\]

or equivalently,

\[
H_{2m,2n+1} \left( \frac{a}{a}, \frac{a}{a} \right) = \left\{ \left( \frac{w - 2}{2m - 1} \right) + 1 \right\} a^{-w} \zeta(w)
\]

\[
+ a^{-w} \sum_{\ell=2}^{2m} \left( \frac{w - \ell - 1}{2n} \right) (-1)^{\ell} \zeta \left( \frac{w - \ell - 1}{2n} \right) \zeta(\ell)
\]

\[
- \left( \frac{w - 2}{2m - 1} \right) H_{1,2m+2n} \left( \frac{a}{a}, \frac{a}{a} \right) + \sum_{\ell=2}^{2n+1} \left( \frac{w - \ell - 1}{2m - 1} \right) H_{w - \ell, \ell} \left( \frac{a}{a}, \frac{a}{a} \right). \tag{33}
\]

Finally we sum together (31) with $b = 1, 2, \ldots, a - 1$ and (33) and get our assertion.

\[\square\]

4. New Sums $K_{p,q}(a,b)$ and $F_{p,q}(k;a,b)$

There is no direct way to evaluate $H_{p,q}(b/a,c/a)$ individually except in a few particular cases. It turns out, however, that some special kinds of linear combinations of $H_{p,q}$’s can be evaluated, which due primarily to the fact that among $H_{p,q}$’s of the same weight there are important relations which are analogous to those among $S_{p,q}$’s of the same weight given by (32).

For simplicity, we shall employ the following notations.

**Notation.** For positive integers $p$, $q$, and $\ell$, we let $w = p + q$ and

\[
A_\ell := A^{p,q}_\ell = (-1)^{\ell} \binom{w - \ell - 1}{q - 1} \quad \text{and} \quad B_\ell := B^{p,q}_\ell = \binom{w - \ell - 1}{p - 1}. \tag{34}
\]

Note that $A_1 = -B_1 = -\binom{w - 2}{p - 1}$.
Proposition 4. For a pair of positive integers $a$ and $b$ with $2b < a$, we let

$$K_{p,q}(a,b) := H_{p,q} \left( \frac{b}{a}, \frac{2b}{a} \right) + H_{p,q} \left( \frac{a-b}{a}, \frac{a-2b}{a} \right).$$

Then for positive integers $p$ and $q$ with $p, q \geq 2$ and $w = p + q$, we have

$$K_{p,q}(a,b) = (-1)^p \sum_{\ell=2}^{q} B_{\ell} K_{w-\ell,\ell}(a,b) + k_{p,q}(a,b),$$

where

$$k_{p,q}(a,b)$$

$$= (-1)^p \sum_{\ell=2}^{p} A_{\ell} a^{-\ell} \left\{ \zeta \left( \ell, \frac{b}{a} \right) \zeta \left( w - \ell, \frac{b}{a} \right) + \zeta \left( \ell, \frac{a-b}{a} \right) \zeta \left( w - \ell, \frac{a-b}{a} \right) \right\}$$

$$+ (-1)^p A_{1} a^{-w} \left\{ \zeta \left( w - 1, \frac{b}{a} \right) - \zeta \left( w - 1, \frac{a-b}{a} \right) \right\} \left\{ \pi \cot \frac{\pi b}{a} - \pi \cot \frac{2\pi b}{a} \right\}$$

$$- (-1)^p A_{1} \sum_{\ell=2}^{p} a^{-\ell} \zeta \left( \ell, \frac{a-b}{a} \right) \zeta \left( w - \ell, \frac{b}{a} \right).$$

Proof. The partial fraction decomposition in (30) can be generalized to

$$\frac{1}{(x + \alpha)^q x^p} = \sum_{\ell=2}^{p} A_{\ell} \frac{(-1)^p}{\alpha^{w-\ell} x^\ell} + \sum_{\ell=2}^{q} B_{\ell} \frac{(-1)^p}{\alpha^{w-\ell} (x + \alpha)^p} + A_{1} \frac{(-1)^p}{\alpha^{w-1}} \left\{ \frac{1}{x} - \frac{1}{x + \alpha} \right\}.$$  

Set $x = aj + b$ and $\alpha = ak + b$ and sum over all non-negative integers $j$ and $k$ to get

$$H_{p,q} \left( \frac{b}{a}, \frac{2b}{a} \right)$$

$$= (-1)^p \sum_{\ell=2}^{q} B_{\ell} H_{w-\ell,\ell} \left( \frac{b}{a}, \frac{2b}{a} \right) + (-1)^p \sum_{\ell=2}^{p} A_{\ell} a^{-w} \zeta \left( \ell, \frac{b}{a} \right) \zeta \left( w - \ell, \frac{b}{a} \right)$$

$$+ (-1)^p A_{1} a^{-w} \zeta \left( w - 1, \frac{b}{a} \right) \left\{ -\psi \left( \frac{b}{a} \right) + \psi \left( \frac{2b}{a} \right) \right\}$$

$$+ (-1)^p A_{1} H_{1,w-1} \left( \frac{2b}{a}, \frac{b}{a} \right).$$
The relation for \( H_{p,q}(\frac{a-b}{a}, \frac{a-2b}{a}) \) can be obtained from the above simply by replacing \( b \) by \( a-b \) and \( 2b \) by \( a-2b \). Therefore, we have

\[
H_{p,q}\left(\frac{a-b}{a}, \frac{a-2b}{a}\right) = (-1)^p \sum_{\ell=2}^{q} B_{\ell} H_{w-\ell,\ell}\left(\frac{a-b}{a}, \frac{a-2b}{a}\right)
+ (-1)^p \sum_{\ell=2}^{p} A_{\ell} a^{-w} \zeta\left(\ell, \frac{a-b}{a}\right) \zeta\left(w-\ell, \frac{a-b}{a}\right)
+ (-1)^p A_{1} a^{-w} \left\{ -\psi\left(\frac{a-b}{a}\right) + \psi\left(\frac{a-2b}{a}\right) \right\}
+ (-1)^p A_{1} H_{1,w-1}\left(\frac{a-2b}{a}, \frac{a-b}{a}\right).
\]

(40)

Our assertion then follows from the combination of (39) and (40), and the evaluation of

\[
H_{1,w-1}\left(\frac{2b}{a}, \frac{b}{a}\right) + H_{1,w-1}\left(\frac{a-2b}{a}, \frac{a-b}{a}\right),
\]

as given in Proposition 2.

In exactly the same manner, we get the following.

**Proposition 5.** For positive integers \( a, b, k \) with \( k \geq 3 \) and \( kb < a \), we let

\[
F_{p,q}(k; a, b) := H_{p,q}\left(\frac{b}{a}, \frac{kb}{a}\right) + H_{p,q}\left(\frac{(k-1)b}{a}, \frac{kb}{a}\right)
+ H_{p,q}\left(\frac{a-b}{a}, \frac{a-kb}{a}\right) + H_{p,q}\left(\frac{a-(k-1)b}{a}, \frac{a-kb}{a}\right).
\]

(41)

Then for \( p, q \geq 2 \) with \( w = p + q \), we have

\[
F_{p,q}(k; a, b) = (-1)^p \sum_{\ell=2}^{q} \binom{w-\ell-1}{p-1} F_{w-\ell,\ell}(k; a, b) + f_{p,q}(k; a, b),
\]

where \( f_{p,q}(k; a, b) \) is a known value in Hurwitz zeta functions.

A few immediate consequences of the previous two propositions are of interest.
Corollary 1. Suppose that $K_{p,q}(a,b)$ is defined as in Proposition 4. Then for any positive integer $n$ and $w = 2n + 3$, we have

$$2K_{2n+1,2}(a,b)$$

$$= \sum_{\ell=2}^{2n+1} (-1)^{\ell+1}(2n + 2 - \ell)$$

$$a^{-w} \left\{ \zeta \left( \frac{\ell}{a}, \frac{b}{a} \right) \zeta \left( w - \ell, \frac{a-b}{a} \right) + \zeta \left( \ell, \frac{a-b}{a} \right) \zeta \left( w - \ell, \frac{a}{a} \right) \right\}$$

$$+ (2n + 1)a^{-w} \left\{ \zeta \left( w - 1, \frac{b}{a} \right) - \zeta \left( w - 1, \frac{a-b}{a} \right) \right\} \{ \pi \cot \frac{\pi b}{a} - \pi \cot \frac{2\pi b}{a} \}$$

$$- (2n + 1) \sum_{\ell=2}^{2n+1} a^{-w} \zeta \left( \ell, \frac{a-b}{a} \right) \zeta \left( w - \ell, \frac{b}{a} \right).$$

When $a = 3$, the new sum

$$K_{p,q}(3,1) = \sum_{k=0}^{\infty} \frac{1}{(3k+1)^p} \sum_{j=0}^{k-1} \frac{1}{(3j+2)^q} + \sum_{k=0}^{\infty} \frac{1}{(3k+2)^p} \sum_{j=0}^{k} \frac{1}{(3j+1)^q}$$

satisfies the reflection formula

$$K_{p,q}(3,1) + K_{q,p}(3,1) = 3^{-(p+q)} \left\{ \zeta \left( \frac{p}{3}, \frac{2}{3} \right) \zeta \left( q, \frac{1}{3} \right) + \zeta \left( \frac{p}{3}, \frac{1}{3} \right) \zeta \left( q, \frac{2}{3} \right) \right\},$$

and $K_{1,3}(3,1)$ can be evaluated for all positive integers $n \geq 2$. Therefore, $K_{p,q}(3,1)$ can be evaluated explicitly like the standard Euler sum $S_{p,q}$ when the weight $p + q$ is odd.

Corollary 2. For odd weight $p + q$ or $(p,q) = (1,n)$ or $(2,4)$ or $(4,2)$, $K_{p,q}(3,1)$ can be evaluated in terms of Hurwitz zeta values.

Corollary 3. For a pair of positive integers $a$ and $b$ with $b < a$, the sum

$$H_{p,q} \left( \frac{b}{a}, \frac{b}{a} \right) + H_{p,q} \left( \frac{a-b}{a}, \frac{a-b}{a} \right) = H_{p,q} \left( \frac{a}{a}, \frac{b}{a} \right) + H_{p,q} \left( \frac{a}{a}, \frac{a-b}{a} \right)$$

$$+ H_{p,q} \left( \frac{b}{a}, \frac{a}{a} \right) + H_{p,q} \left( \frac{a-b}{a}, \frac{a}{a} \right)$$

can be evaluated when $p + q$ is odd or $(p,q) = (1,n)$ or $(2,4)$ or $(4,2)$.

5. Evaluation of $K_{p,q}(a,b)$ and $F_{p,q}(k;a,b)$ of Odd Weight

Recall that for positive integers $p, q, a, b$ with $q \geq 2$ and $2b < a$,

$$K_{p,q}(a,b) = H_{p,q} \left( \frac{b}{a}, \frac{2b}{a} \right) + H_{p,q} \left( \frac{a-b}{a}, \frac{a-2b}{a} \right).$$

To evaluate $K_{p,q}$, it is necessary to introduce its companion families. Define

$$L_{p,q}(a,b) := H_{p,q} \left( \frac{2b}{a}, \frac{b}{a} \right) + H_{p,q} \left( \frac{a-2b}{a}, \frac{a-b}{a} \right).$$
and

\[ M_{p,q}(a,b) := H_{p,q} \left( \frac{b}{a}, \frac{a-b}{a} \right) + H_{p,q} \left( \frac{a-b}{a}, b \right). \]  

(46)

From the above definitions, we see immediately from (9) the following reflection formula: For \( p, q \in \mathbb{Z}^+ \) with \( p, q \geq 2 \),

\[ K_{p,q}(a,b) + L_{q,p}(a,b) = a^{-(p+q)} \{ \zeta \left( p, \frac{b}{a} \right) \zeta \left( q, \frac{2b}{a} \right) + \zeta \left( p, \frac{a-b}{a} \right) \zeta \left( q, \frac{a-2b}{a} \right) \}. \]  

(47)

\[ M_{p,q}(a,b) + M_{q,p}(a,b) = a^{-(p+q)} \{ \zeta \left( p, \frac{b}{a} \right) \zeta \left( q, \frac{a-b}{a} \right) + \zeta \left( p, \frac{a-b}{a} \right) \zeta \left( q, \frac{b}{a} \right) \}. \]  

(48)

These three families are delicately interwoven together and satisfy the same kind of relations as those among \( E_{p,q}^{(a)}, T_{p,q}^{(a)} \) and \( D_{p,q}^{(a)} \), which we now describe.

By a similar consideration as in Proposition 4, we get

\[ K_{p,q}(a,b) = (-1)^p \sum_{\ell=2}^{q} B_{2\ell} K_{w-\ell,\ell}(a,b) + (-1)^p A_1 L_{1,w-1}(a,b) + k_1(p,q,a,b), \]  

(49)

\[ L_{p,q}(a,b) = (-1)^p \sum_{\ell=2}^{q} B_{2\ell} M_{w-\ell,\ell}(a,b) + (-1)^p A_1 M_{1,w-1}(a,b) + k_2(p,q,a,b), \]  

(50)

and

\[ M_{p,q}(a,b) = (-1)^p \sum_{\ell=2}^{q} B_{2\ell} L_{w-\ell,\ell}(a,b) + (-1)^p A_1 K_{1,w-1}(a,b) + k_3(p,q,a,b), \]  

(51)

where \( k_j(p,q,a,b), j = 1, 2, 3, \) are known values in Hurwitz zeta functions. More precisely,

\[ k_2(p,q,a,b) = (-1)^p \sum_{\ell=2}^{q} A_{\ell} a^{-w} \left\{ \zeta \left( \ell, \frac{2b}{a} \right) \zeta \left( w-\ell, \frac{a-b}{a} \right) + \zeta \left( \ell, \frac{a-2b}{a} \right) \zeta \left( w-\ell, \frac{b}{a} \right) \right\} + (-1)^p A_1 a^{-w} \left\{ \zeta \left( w-1, \frac{a-b}{a} \right) \left( -\psi \left( \frac{2b}{a} \right) + \psi \left( \frac{b}{a} \right) \right) \right. \]  

\[ + \zeta \left( w-1, \frac{b}{a} \right) \left( -\psi \left( \frac{a-2b}{a} \right) + \psi \left( \frac{a-b}{a} \right) \right) \left\} \right. \]  

(52)

and

\[ k_3(p,q,a,b) = (-1)^p \sum_{\ell=2}^{q} A_{\ell} a^{-w} \left\{ \zeta \left( \ell, \frac{b}{a} \right) \zeta \left( w-\ell, \frac{a-2b}{a} \right) + \zeta \left( \ell, \frac{a-b}{a} \right) \zeta \left( w-\ell, \frac{2b}{a} \right) \right\} + (-1)^p A_1 a^{-w} \left\{ \left( -\zeta \left( w-1, \frac{a-2b}{a} \right) - \zeta \left( w-1, \frac{2b}{a} \right) \right) \right. \]  

\[ \left. \pi \cot \left( \frac{b\pi}{a} \right) \right\}. \]  

(53)
Such kind of relations are precisely the same relations among $E_{p,q}^{(a)}$, $T_{p,q}^{(a)}$ and $D_{p,q}^{(a)}$. Therefore when the weight $p + q$ is odd, we can solve for $K_{p,q}(a,b)$, $L_{p,q}(a,b)$ and $M_{p,q}(a,b)$ from these relations provided that we know the values of $K_{1,2n}(a,b)$, $L_{1,2n}(a,b)$ and $M_{1,2n}(a,b)$ in advance. To be precise, let us reinterpret the above relations in matrix form. Suppose $w = p + q = 2n + 1$ for $n \geq 2$. Let $r = w - 3 = 2n - 2$ and define the $r \times r$ matrix $B = (B_{p,\ell})_{2 \leq p, \ell \leq w-2}$ by

$$B_{p,\ell} := (-1)^p B_{w-\ell}^{p,q} = (-1)^p \left( \frac{\ell - 1}{p - 1} \right),$$

for $p, \ell = 2, 3, \ldots, 2n - 1$. Thus, for example, $B = \begin{bmatrix} 1 & -1 \end{bmatrix}$ for $w = 5$; while for $w = 7$,

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -6 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$  

By abuse of notation, let $K, L, M$ denote, respectively, the column vectors consisting of the three sets of variables $K_{2,w-2}, K_{3,w-3}, \ldots, K_{w-2,2}$; $L_{2,w-2}, L_{3,w-3}, \ldots, L_{w-2,2}$; and $M_{2,w-2}, M_{3,w-3}, \ldots, M_{w-2,2}$. Then the above relations (47), (48), (49), (50), and (51), with $p$ varying from 2 to $2n - 1$, give rise to five linear systems, respectively:

$$K + JL = c,$$  

$$\left( I + J \right)M = d,$$  

$$K = BK + k,$$  

$$L = BM + l,$$  

$$M = BL + m,$$

where $I$ is the $r \times r$ identity matrix, $J$ is a permutation matrix with 1 on the reverse diagonal and 0 elsewhere,

$$J = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

and $c, d, k, l, m$ are column $r$ vectors consisting of known values. Note that $K, L, M$ are determined if any one of them is determined. However, the system (3) alone does not uniquely determine $K$ since the rank of the matrix $I - B$ is only $r/2$. A remedy is readily given by the self-reflection formula (see (48) and (2) above) enjoyed by $M$. This means that one can unwind $M$ first because (2) provides the extra $r/2$ relations.

From (1) and (4), one has

$$K = c - JL = -JBM + (c - JL).$$  

Substituting (6) into (3), we get

$$(B - I)JBM = (B - I)(c - JL) + k.$$
The system (7) is still not sufficient to determine $M$ since the rank of $(B - I)JB$ is $n - 1 = r/2 < r$. This can be resolved by considering systems (2) and (7) together. Subtracting (2) from (7), we get $PM = b$, where

$$P = (B - I)JB - (I + J)$$

and

$$b = (B - I)(c - JI) + k - d.$$ 

Now, we shall show that $P$ is invertible, and furthermore, that the minimal polynomial of $P$ is $x^2 + 4x + 4$ if $w = 5$, and is $x^3 + 3x^2 - 4$ for $w = 2n + 1 \geq 7$. Our proof is elementary and hinges on the following known combinatorial identity.

**Lemma.** For $0 \leq \mu, \nu \leq m$,

$$\sum_{k=0}^{m} (-1)^{k} \binom{m-k}{\mu} \binom{\mu}{k} = \binom{m-\mu}{m-\nu}.$$ 

In particular, setting $\mu = m$,

$$\sum_{k=0}^{m} (-1)^{k} \binom{m-k}{\nu} \binom{m}{k} = (-1)^{m} \delta_{m\nu}.$$ 

Denote by $(A)_{ij}$ the $(i, j)$ entry of a matrix $A$. Note that $(J)_{ij} = \delta_{i+j,r+1}$, where $\delta$ is the Kronecker delta, and $(B)_{ij} = (-1)^{i+1} \binom{r}{j}$). An easy calculation shows that $J^2 = B^2 = I$,

$$(JB)_{ij} = (-1)^{r-i} \binom{j}{r+1-i} = (-1)^i \binom{j}{r+1-i},$$

and

$$(BJ)_{ij} = (J(B)J)_{ij} = (-1)^{r+1-i} \binom{r+1-j}{i} = (-1)^{i+1} \binom{r+1-j}{i}.$$ 

Using the lemma above,

$$((JB)^2)_{ij} = (-1)^r \sum_{k=1}^{r} (-1)^{k} \binom{k}{r+1-i} \binom{j}{r+1-k}$$

$$= (-1)^{i+1} \sum_{k=1}^{r} (-1)^{k} \binom{r+1-k}{r+1-i} \binom{j}{k}$$

$$= (-1)^{i+1} \left\{ \binom{r+1-j}{i} - \binom{r+1}{i} \right\}.$$ 

Hence,

$$((BJ)^2)_{ij} = (J(BJ)^2J)_{ij} = (-1)^{r-i} \left\{ \binom{j}{r+1-i} - \binom{r+1}{r+1-i} \right\}$$

$$= (-1)^i \left\{ \binom{j}{r+1-i} - \binom{r+1}{i} \right\},$$

and it follows that $(JB)^2 + (BJ)^2 = JB + BJ$. Using this relation together with $J^2 = B^2 = I$, a straightforward, though somehow tedious, calculation then confirms
the minimal polynomial for $P$ as given above. In any circumstance, one has $P^{-1} = (P^2 + 3P)/4$, and therefore the solution of $M$ is given by the recipe

$$M = P^{-1}b = \frac{1}{4}(P^2 + 3P)((c - J) + k - d).$$

The algebraic formalism described above enables one to derive explicit evaluation of $K_{p,q}(a, b)$ when the weight $p + q$ is odd. The first step is to evaluate the three families $K_{p,q}, L_{p,q}$, and $M_{p,q}$ at index $p = 1$. $L_{1,n}$ can be evaluated for all positive integers $n \geq 2$ via Proposition 2. Here we determine $K_{1,2n}$ and $M_{1,2n}$ from the above relations.

**Proposition 6.** For each positive integer $n$, $K_{1,2n}(a, b)$ and $M_{1,2n}(a, b)$ can be evaluated explicitly in terms of Hurwitz zeta values.

**Proof.** Note that

$$K_{1,2n}(a, b) + M_{1,2n}(a, b) = \left\{ H_{1,2n} \left( \frac{b}{a}, \frac{2b}{a} \right) + H_{1,2n} \left( \frac{a - b}{a}, \frac{b}{a} \right) \right\}$$

$$+ \left\{ H_{1,2n} \left( \frac{b}{a}, \frac{a - b}{a} \right) + H_{1,2n} \left( \frac{a - b}{a}, \frac{a - 2b}{a} \right) \right\}.$$  \hspace{1cm} (54)

According to Proposition 2, the value is equal to

$$a^{-w} \zeta \left( 2n, \frac{b}{a} \right) \left\{ -\psi \left( \frac{b}{a} \right) + \psi \left( \frac{2b}{a} \right) \right\}$$

$$+ a^{-w} \zeta \left( 2n, \frac{a - b}{a} \right) \left\{ -\psi \left( \frac{a - b}{a} \right) + \psi \left( \frac{a - 2b}{a} \right) \right\}$$

$$+ a^{-w} \left\{ \zeta \left( 2n, \frac{a - 2b}{a} \right) - \zeta \left( 2n, \frac{2b}{a} \right) \right\} \pi \cot \frac{b\pi}{a}$$

$$- \sum_{\ell = 2}^{2n - 1} a^{-w} \left\{ \zeta \left( \ell, \frac{2b}{a} \right) \zeta \left( 2n + 1 - \ell, \frac{b}{a} \right) + \zeta \left( \ell, \frac{a - b}{a} \right) \zeta \left( 2n + 1 - \ell, \frac{a - 2b}{a} \right) \right\}. \hspace{1cm} (55)$$

On the other hand, set $q = 2$ and $p = 2n - 1$ in (50) and (51) to yield

$$L_{2n-1,2}(a,b) = -M_{2n-1,2}(a,b) + (w - 2)M_{1,2n}(a,b) + k_2(2n - 1, 2, a, b)$$

and

$$M_{2n-1,2}(a,b) = -L_{2n-1,2}(a,b) + (w - 2)K_{1,2n}(a,b) + k_3(2n - 1, 2, a, b).$$

It follows that

$$(w - 2) \{K_{1,2n}(a,b) - M_{1,2n}(a,b)\} = k_2(2n - 1, 2, a, b) - k_3(2n - 1, 2, a, b). \hspace{1cm} (56)$$

Our assertion then follows from (54) and (56).

**Theorem 4.** Suppose that $K_{p,q}(a,b)$, $L_{p,q}(a,b)$ and $M_{p,q}(a,b)$ are defined as in (44), (45) and (46) for positive integers $p, q, a, b$ with $q \geq 2$ and $2b < a$. Then they can be evaluated in terms of Hurwitz zeta values.
For the family
\[
F_{p,q}(k; a, b) = H_{p,q} \left( \frac{b}{a}, \frac{kb}{a} \right) + H_{p,q} \left( \frac{(k-1)b}{a}, \frac{kb}{a} \right) \\
+ H_{p,q} \left( \frac{a - b}{a}, \frac{a - kb}{a} \right) + H_{p,q} \left( \frac{a - (k-1)b}{a}, \frac{a - kb}{a} \right),
\]
we introduce another two families
\[
G_{p,q}(k; a, b) := H_{p,q} \left( \frac{kb}{a}, \frac{b}{a} \right) + H_{p,q} \left( \frac{kb}{a}, \frac{(k-1)b}{a} \right) \\
+ H_{p,q} \left( \frac{a - kb}{a}, \frac{a - b}{a} \right) + H_{p,q} \left( \frac{a - kb}{a}, \frac{a - (k-1)b}{a} \right),
\]
and
\[
R_{p,q}(k; a, b) := H_{p,q} \left( \frac{b}{a}, \frac{a - (k-1)b}{a} \right) + H_{p,q} \left( \frac{(k-1)b}{a}, \frac{a - b}{a} \right) \\
+ H_{p,q} \left( \frac{a - b}{a}, \frac{(k-1)b}{a} \right) + H_{p,q} \left( \frac{a - (k-1)b}{a}, \frac{b}{a} \right).
\]
Again \(F_{p,q}(k; a, b), G_{p,q}(k; a, b)\) and \(R_{p,q}(k; a, b)\) satisfy relations like (49), (50) and (51) and possess reflection formulæ we need. Consequently we have the following theorem.

**Theorem 5.** Suppose that \(F_{p,q}(k; a, b), G_{p,q}(k; a, b)\) and \(R_{p,q}(k; a, b)\) are defined in (57), (58) and (59) for positive integers \(p, q, a, b\) with \(q \geq 2\) and \(kb < a\), \(k \geq 3\). Then they can be evaluated in terms of Hurwitz zeta values when the weight \(w = p + q\) is odd.

As final remarks, we first mention that exactly the same argument may be applied to the family
\[
H_{p,q} \left( \frac{jb}{a}, \frac{kb}{a} \right) + H_{p,q} \left( \frac{(k-j)b}{a}, \frac{kb}{a} \right) \\
+ H_{p,q} \left( \frac{a - jb}{a}, \frac{a - kb}{a} \right) + H_{p,q} \left( \frac{a - (k-j)b}{a}, \frac{a - kb}{a} \right),
\]
and explicit evaluations may therefore be obtained. As for the case of even weight, and more generalized Euler sums with Dirichlet characters, some extra complication arises and we shall defer the discussion in a later paper [5].

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References