GENERALIZED POPOVICIU FUNCTIONAL EQUATIONS IN BANACH MODULES OVER A $C^*$-ALGEBRA AND APPROXIMATE ALGEBRA HOMOMORPHISMS

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1. Introduction

Let $E_1$ and $E_2$ be Banach spaces with norms $\| \cdot \|$ and $\| \cdot \|$, respectively. Consider $f : E_1 \to E_2$ to be a mapping such that $f(t x)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\varepsilon \geq 0$ and $p \in [0,1)$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon (\|x\|^p + \|y\|^p)$$

for all $x, y \in E_1$. Th. M. Rassias [11] showed that there exists a unique $\mathbb{R}$–linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\varepsilon}{2 - 2p} \|x\|^p$$

for all $x \in E_1$.

Throughout this paper, let $A$ be a unital $C^*$–algebra with norm $\| \cdot \|$, $U(A)$ the unitary group of $A$, $A_{in}$ the set of invertible elements in $A$, $A_{sa}$ the set of self-adjoint elements in $A$, $A_1 = \{a \in A \mid \|a\| = 1\}$, and $A_1^+$ the set of positive elements in $A_1$. Let $_AB$ and $_AC$ be left Banach $A$–modules with norms $\| \cdot \|$ and $\| \cdot \|$, respectively. Let $n$ and $k$ be integers with $2 \leq k \leq n - 1$.

Recently, T. Trif [19] generalized the Popoviciu functional equation

$$3f \left( \frac{x + y + z}{3} \right) + f(x) + f(y) + f(z) = 2 \left( f \left( \frac{x + y}{2} \right) + f \left( \frac{y + z}{2} \right) + f \left( \frac{z + x}{2} \right) \right)$$

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Lemma A. [19, Theorem 2.1] Let $V$ and $W$ be vector spaces. A mapping $f : V \to W$ with $f(0) = 0$ satisfies the functional equation

$$n \cdot n-2C_{k-2}f\left(\frac{x_1 + \cdots + x_n}{n}\right) + n-2C_{k-1} \sum_{i=1}^{n} f(x_i) = k \cdot \sum_{1 \leq i_1 < \cdots < i_k \leq n} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right) \quad (A)$$

for all $x_1, \cdots, x_n \in V$ if and only if the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x + y) = f(x) + f(y)$ for all $x, y \in V$.

The following is useful to prove the stability of the functional equation $(A)$.

Lemma B. [10, Theorem 1] Let $a \in A$ and $|a| < 1 - \frac{2}{m}$ for some integer $m$ greater than 2. Then there are $m$ elements $u_1, \cdots, u_m \in \mathcal{U}(A)$ such that $ma = u_1 + \cdots + u_m$.

The main purpose of this paper is to prove the Hyers–Ulam–Rassias stability of the functional equation $(A)$ in Banach modules over a unital $C^*$-algebra, and to prove the Hyers–Ulam–Rassias stability of algebra homomorphisms between Banach algebras associated with the functional equation $(A)$.

2. Stability of Generalized Popoviciu Functional Equations in Banach Modules over a $C^*$-Algebra Associated with its Unitary Group

We are going to prove the Hyers–Ulam–Rassias stability of the functional equation $(A)$ in Banach modules over a unital $C^*$-algebra associated with its unitary group.

For a given mapping $f : A\mathcal{B} \to A\mathcal{C}$ and a given $a \in A$, we set

$$D_a f(x_1, \cdots, x_n) := n \cdot n-2C_{k-2}af\left(\frac{x_1 + \cdots + x_n}{n}\right) + n-2C_{k-1} \sum_{i=1}^{n} a f(x_i) - k \cdot \sum_{1 \leq i_1 < \cdots < i_k \leq n} f\left(\frac{ax_{i_1} + \cdots + ax_{i_k}}{k}\right)$$

for all $x_1, \cdots, x_n \in A\mathcal{B}$.

**Theorem 2.1.** Let $q = \frac{k(n-1)}{n-k}$ and $r = -\frac{k}{n-k}$. Let $f : A\mathcal{B} \to A\mathcal{C}$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : A\mathcal{B}^n \to [0, \infty)$ such that

$$\varphi(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} q^j \varphi(q^j x_1, \cdots, q^j x_n)$$

$$< \infty \|D_u f(x_1, \cdots, x_n)\| \leq \varphi(x_1, \cdots, x_n) \quad (2.i)$$

for all $u \in \mathcal{U}(A)$ and all $x_1, \cdots, x_n \in A\mathcal{B}$. Then there exists a unique $A$-linear mapping $T : A\mathcal{B} \to A\mathcal{C}$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{k \cdot n-1C_{k-1}} \varphi(qx, rx, \cdots, rx) \quad (2.ii)$$

for all $x \in A\mathcal{B}$. 
**Proof.** Put $u = 1 \in U(A)$. By [19, Theorem 3.1], there exists a unique additive mapping $T : A \mathbb{B} \to A \mathbb{C}$ satisfying (2.ii). The additive mapping $T : A \mathbb{B} \to A \mathbb{C}$ was defined by

$$T(x) = \lim_{j \to \infty} q^{-j} f(q^j x)$$

for all $x \in A \mathbb{B}$.

By the assumption, for each $u \in U(A)$,

$$q^{-j} \|Du f(q^j x, \ldots, q^j x)\| \leq q^{-j} \varphi(q^j x, \ldots, q^j x)$$

for all $x \in A \mathbb{B}$, and

$$q^{-j} \|Du f(q^j x, \ldots, q^j x)\| \to 0$$

as $n \to \infty$ for all $x \in A \mathbb{B}$. So

$$Du T(x, \ldots, x) = \lim_{j \to \infty} q^{-j} Du f(q^j x, \ldots, q^j x) = 0$$

for all $u \in U(A)$ and all $x \in A \mathbb{B}$. Hence

$$Du T(x, \ldots, x) = n \cdot n - 2 C_{k - 2} u T(x) + n \cdot n - 2 C_{k - 1} u T(x) - n \cdot n C_k T(u x) = 0$$

for all $u \in U(A)$ and all $x \in A \mathbb{B}$. So

$$u T(x) = T(u x)$$

for all $u \in U(A)$ and all $x \in A \mathbb{B}$.

Now let $a \in A$ ($a \neq 0$) and $M$ an integer greater than $4|a|$. Then

$$\left| \frac{a}{M} \right| = \frac{1}{M} |a| < \frac{|a|}{4|a|} = \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3} .$$

By Lemma B, there exist three elements $u_1, u_2, u_3 \in U(A)$ such that $3 \frac{a}{M} = u_1 + u_2 + u_3$. And $T(x) = T \left( 3 \cdot \frac{1}{3} x \right) = 3T \left( \frac{1}{3} x \right)$ for all $x \in A \mathbb{B}$. So $T \left( \frac{1}{3} x \right) = \frac{1}{3} T(x)$ for all $x \in A \mathbb{B}$. Thus

$$T(ax) = T \left( \frac{M}{3} \cdot 3 \frac{a}{M} x \right) = M \cdot T \left( \frac{1}{3} \cdot 3 \frac{a}{M} x \right) = \frac{M}{3} T \left( 3 \frac{a}{M} x \right)$$

$$= \frac{M}{3} T(u_1 x + u_2 x + u_3 x) = \frac{M}{3} \left( T(u_1 x) + T(u_2 x) + T(u_3 x) \right)$$

$$= \frac{M}{3} (u_1 + u_2 + u_3) T(x) = \frac{M}{3} \cdot \frac{3}{M} T(x)$$

$$= a T(x)$$

for all $x \in A \mathbb{B}$. Obviously, $T(0x) = 0T(x)$ for all $x \in A \mathbb{B}$. Hence

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in A \mathbb{B}$. So the unique additive mapping $T : A \mathbb{B} \to A \mathbb{C}$ is an $A$–linear mapping, as desired.

□

Applying the unital $C^*$–algebra $\mathbb{C}$ to Theorem 2.1, one can obtain the following.
Corollary 2.2. Let $E_1$ and $E_2$ be complex Banach spaces with norms $\| \cdot \|$ and $\| \cdot \|$, respectively. Let $q = \frac{k(n-1)}{n-k}$ and $r = -\frac{k}{n-k}$. Let $f : E_1 \to E_2$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : E_1^n \to [0, \infty)$ such that

$$
\varphi(x_1, \ldots, x_n) := \sum_{j=0}^{\infty} q^{-j} \varphi(q^j x_1, \ldots, q^j x_n)
$$

for all $\lambda \in T^1 := \{ \lambda \in \mathbb{C} \mid |\lambda| = 1 \}$ and all $x_1, \ldots, x_n \in E_1$. Then there exists a unique $\mathbb{C}$-linear mapping $T : E_1 \to E_2$ such that

$$
\| f(x) - T(x) \| < \infty \| D_\lambda f(x_1, \ldots, x_n) \| \leq \varphi(x_1, \ldots, x_n)
$$

for all $x \in E_1$.

Theorem 2.3. Let $q = \frac{k(n-1)}{n-k}$ and $r = -\frac{k}{n-k}$. Let $f : A \mathbb{B} \to A \mathbb{C}$ be a continuous mapping with $f(0) = 0$ for which there exists a function $\varphi : A \mathbb{B}^n \to [0, \infty)$ satisfying (2.i) such that

$$
\| D_u f(x_1, \ldots, x_n) \| \leq \varphi(x_1, \ldots, x_n)
$$

for all $u \in U(A)$ and all $x_1, \ldots, x_n \in A \mathbb{B}$. If the sequence $\{ q^{-j} f(q^j x) \}$ converges uniformly on $A \mathbb{B}$, then there exists a unique continuous $A$-linear mapping $T : A \mathbb{B} \to A \mathbb{C}$ satisfying (2.ii).

Proof. Put $u = 1 \in U(A)$. By Theorem 2.1, there exists a unique $A$-linear mapping $T : A \mathbb{B} \to A \mathbb{C}$ satisfying (2.ii). By the continuity of $f$, the uniform convergence and the definition of $T$, the $A$-linear mapping $T : A \mathbb{B} \to A \mathbb{C}$ is continuous, as desired. \hfill \Box

Theorem 2.4. Let $q = \frac{k(n-1)}{n-k}$ and $r = -\frac{1}{n-k}$. Let $f : A \mathbb{B} \to A \mathbb{C}$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : A \mathbb{B}^n \to [0, \infty)$ such that

$$
\varphi(x_1, \ldots, x_n) := \sum_{j=0}^{\infty} q^{-j} \varphi(q^{-j} x_1, \ldots, q^{-j} x_n) < \infty
$$

for all $u \in U(A)$ and all $x_1, \ldots, x_n \in A \mathbb{B}$. Then there exists a unique $A$-linear mapping $T : A \mathbb{B} \to A \mathbb{C}$ such that

$$
\| f(x) - T(x) \| \leq \frac{1}{n-2C_k-1} \varphi(x, rx, \ldots, rx)
$$

for all $x \in A \mathbb{B}$.

Proof. Put $u = 1 \in U(A)$. By [19, Theorem 3.2], there exists a unique additive mapping $T : A \mathbb{B} \to A \mathbb{C}$ satisfying (2.iv). The additive mapping $T : A \mathbb{B} \to A \mathbb{C}$ was defined by

$$
T(x) = \lim_{j \to \infty} q^j f(q^{-j} x)
$$

for all $x \in A \mathbb{B}$.

By the assumption, for each $u \in U(A)$,

$$
q^j \| D_u f(q^{-j} x, \ldots, q^{-j} x) \| \leq q^j \varphi(q^{-j} x, \ldots, q^{-j} x)
$$

for all \( x \in \mathcal{A} \mathcal{B} \), and

\[
q^j \| D_u f(q^{-j} x, \cdots, q^{-j} x) \| \rightarrow 0
\]
as \( n \rightarrow \infty \) for all \( x \in \mathcal{A} \mathcal{B} \). So

\[
D_u T(x, \cdots, x) = \lim_{j \rightarrow \infty} q^j D_u f(q^{-j} x, \cdots, q^{-j} x) = 0
\]
for all \( u \in \mathcal{U}(A) \) and all \( x \in \mathcal{A} \mathcal{B} \). Hence

\[
D_u T(x, \cdots, x) = n \cdot n_{-2} C_{k-2} u T(x) + n \cdot n_{-2} C_{k-1} u T(x) - k \cdot n C_k T(u x) = 0
\]
for all \( u \in \mathcal{U}(A) \) and all \( x \in \mathcal{A} \mathcal{B} \). So

\[
u T(x) = T(u x)
\]
for all \( u \in \mathcal{U}(A) \) and all \( x \in \mathcal{A} \mathcal{B} \).

The rest of the proof is the same as the proof of Theorem 2.1. \( \Box \)


Given a locally compact abelian group \( G \) and a multiplier \( \omega \) on \( G \), one can associate to them the twisted group \( C^* \)-algebra \( C^*(G, \omega) \). \( C^*(\mathbb{Z}^m, \omega) \) is said to be a noncommutative torus of rank \( m \) and denoted by \( A_\omega \). The multiplier \( \omega \) determines a subgroup \( S_\omega \) of \( G \), called its symmetry group, and the multiplier is called totally skew if the symmetry group \( S_\omega \) is trivial. And \( A_\omega \) is called completely irrational if \( \omega \) is totally skew (see [2]). It was shown in [2] that if \( G \) is a locally compact abelian group and \( \omega \) is a totally skew multiplier on \( G \), then \( C^*(G, \omega) \) is a simple \( C^* \)-algebra. It was shown in [3, Theorem 1.5] that if \( A_\omega \) is a simple noncommutative torus then \( A_\omega \) has real rank \( 0 \), where “real rank \( 0 \)” means that the set of invertible self-adjoint elements is dense in the set of self-adjoint elements (see [3], [6]).

From now on, assume that \( \varphi : \mathcal{A} \mathcal{B}^n \rightarrow [0, \infty) \) is a function satisfying (2.1), and that \( q = \frac{k(n-1)}{n-k} \) and \( r = \frac{k}{n-k} \).

We are going to prove the Hyers–Ulam–Rassias stability of the functional equation (A) in Banach modules over a unital \( C^* \)-algebra.

**Theorem 3.1.** Let \( q \) be not an integer. Let \( f : \mathcal{A} \mathcal{B} \rightarrow \mathcal{A} \mathcal{C} \) be a mapping with \( f(0) = 0 \) such that

\[
\| D_1 f(x_1, \cdots, x_n) \| \leq \varphi(x_1, \cdots, x_n)
\]

for all \( x_1, \cdots, x_n \in \mathcal{A} \mathcal{B} \). Then there exists a unique additive mapping \( T : \mathcal{A} \mathcal{B} \rightarrow \mathcal{A} \mathcal{C} \) satisfying (2.ii). Further, if \( f(\lambda x) \) is continuous in \( \lambda \in \mathbb{R} \) for each fixed \( x \in \mathcal{A} \mathcal{B} \), then the additive mapping \( T : \mathcal{A} \mathcal{B} \rightarrow \mathcal{A} \mathcal{C} \) is \( \mathbb{R} \)-linear.

**Proof.** By the same reasoning as the proof of Theorem 2.1, there exists a unique additive mapping \( T : \mathcal{A} \mathcal{B} \rightarrow \mathcal{A} \mathcal{C} \) satisfying (2.ii).

Assume that \( f(\lambda x) \) is continuous in \( \lambda \in \mathbb{R} \) for each fixed \( x \in \mathcal{A} \mathcal{B} \). By the assumption, \( q \) is a rational number which is not an integer. The additive mapping \( T \) given above is similar to the additive mapping \( T \) given in the proof of [11, Theorem]. By the same reasoning as the proof of [11, Theorem], the additive mapping \( T : \mathcal{A} \mathcal{B} \rightarrow \mathcal{A} \mathcal{C} \) is \( \mathbb{R} \)-linear. \( \Box \)
Theorem 3.2. Let \( q \) be not an integer. Let \( f : \mathcal{A} \mathcal{B} \rightarrow \mathcal{A} \mathcal{C} \) be a continuous mapping with \( f(0) = 0 \) such that

\[
\|D_a f(x_1, \cdots, x_n)\| \leq \varphi(x_1, \cdots, x_n)
\]

for all \( a \in \mathcal{A}^+_1 \cup \{i\} \) and all \( x_1, \cdots, x_n \in \mathcal{A} \mathcal{B} \). If the sequence \( \{q^{-j} f(q^j x)\} \) converges uniformly on \( \mathcal{A} \mathcal{B} \), then there exists a unique continuous \( \mathcal{A} \)-linear mapping \( T : \mathcal{A} \mathcal{B} \rightarrow \mathcal{A} \mathcal{C} \) satisfying (2.ii).

Proof. Put \( a = 1 \in \mathcal{A}^+_1 \). By Theorem 3.1, there exists a unique \( \mathbb{R} \)-linear mapping \( T : \mathcal{A} \mathcal{B} \rightarrow \mathcal{A} \mathcal{C} \) satisfying (2.ii). By the continuity of \( f \) and the uniform convergence, the \( \mathbb{R} \)-linear mapping \( T : \mathcal{A} \mathcal{B} \rightarrow \mathcal{A} \mathcal{C} \) is continuous.

By the same reasoning as the proof of Theorem 2.1,

\[
T(ax) = aT(x)
\]

for all \( a \in \mathcal{A}^+_1 \cup \{i\} \).

For any element \( a \in \mathcal{A} \), \( a = \frac{a+a^*}{2} + i \frac{a-a^*}{2i} \), and \( \frac{a+a^*}{2} \) and \( \frac{a-a^*}{2i} \) are self-adjoint elements, furthermore, \( a = \left( \frac{a+a^*}{2} \right)^+ - \left( \frac{a+a^*}{2} \right)^- + i \left( \frac{a-a^*}{2i} \right)^+ - i \left( \frac{a-a^*}{2i} \right)^- \), where \( \left( \frac{a+a^*}{2} \right)^+ \), \( \left( \frac{a+a^*}{2} \right)^- \), \( \left( \frac{a-a^*}{2i} \right)^+ \), and \( \left( \frac{a-a^*}{2i} \right)^- \) are positive elements (see [4, Lemma 38.8]). So

\[
T(ax) = T \left( \left( \frac{a+a^*}{2} \right)^+ x - \left( \frac{a+a^*}{2} \right)^- x + i \left( \frac{a-a^*}{2i} \right)^+ x - i \left( \frac{a-a^*}{2i} \right)^- x \right)
\]

\[
= \left( \frac{a+a^*}{2} \right)^+ T(x) + \left( \frac{a+a^*}{2} \right)^- T(-x) + i \left( \frac{a-a^*}{2i} \right)^+ T(ix)
\]

\[
+ \left( \frac{a-a^*}{2i} \right)^- T(-ix)
\]

\[
= \left( \frac{a+a^*}{2} \right)^+ T(x) - \left( \frac{a+a^*}{2} \right)^- T(x) + i \left( \frac{a-a^*}{2i} \right)^+ T(x)
\]

\[
- i \left( \frac{a-a^*}{2i} \right)^- T(x)
\]

\[
= \left( \left( \frac{a+a^*}{2} \right)^+ - \left( \frac{a+a^*}{2} \right)^- + i \left( \frac{a-a^*}{2i} \right)^+ - i \left( \frac{a-a^*}{2i} \right)^- \right) T(x)
\]

\[
= aT(x)
\]

for all \( a \in \mathcal{A} \) and all \( x \in \mathcal{A} \mathcal{B} \). Hence

\[
T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)
\]

for all \( a, b \in \mathcal{A} \) and all \( x, y \in \mathcal{A} \mathcal{B} \), as desired. \( \square \)

Theorem 3.3. Let \( \mathcal{A} \) be a unital \( C^* \)-algebra of real rank 0, and \( q \) not an integer. Let \( f : \mathcal{A} \mathcal{B} \rightarrow \mathcal{A} \mathcal{C} \) be a continuous mapping with \( f(0) = 0 \) such that

\[
\|D_a f(x_1, \cdots, x_n)\| \leq \varphi(x_1, \cdots, x_n)
\]
for all \( a \in (A_1^+ \cap A_{in}) \cup \{i\} \) and all \( x_1, \ldots, x_n \in A B \). If the sequence \( \{q^{-i} f(q^j x)\} \) converges uniformly on \( A B \), then there exists a unique continuous \( A \)-linear mapping \( T : A B \to A C \) satisfying (2.ii).

**Proof.** By the same reasoning as the proof of Theorem 3.2, there exists a unique continuous \( \mathbb{R} \)-linear mapping \( T : A B \to A C \) satisfying (2.ii), and

\[
T(ax) = aT(x) \tag{3.1}
\]

for all \( a \in (A_1^+ \cap A_{in}) \cup \{i\} \) and all \( x \in A B \).

Let \( b \in A_1^+ \setminus A_{in} \). Since \( A_{in} \cap A_{sa} \) is dense in \( A_{sa} \), there exists a sequence \( \{b_m\} \) in \( A_{in} \cap A_{sa} \) such that \( b_m \to b \) as \( m \to \infty \). Put \( c_m = \frac{1}{|b_m|} b_m \). Then \( c_m \to \frac{1}{|b|} b = b \) as \( m \to \infty \). Put \( a_m = \sqrt{c_m} c_m \). Then \( a_m \to b \) as \( m \to \infty \) and \( a_m \in A_1^+ \cap A_{in} \). Thus there exists a sequence \( \{a_m\} \) in \( A_1^+ \cap A_{in} \) such that \( a_m \to b \) as \( m \to \infty \), and by the continuity of \( T \)

\[
\lim_{m \to \infty} T(a_m x) = T(\lim_{m \to \infty} a_m x) = T(bx) \tag{3.2}
\]

for all \( x \in A B \). By (3.1),

\[
\|T(a_m x) - bT(x)\| = \|a_m T(x) - bT(x)\| \to \|bT(x) - bT(x)\| = 0 \tag{3.3}
\]

as \( m \to \infty \). By (3.2) and (3.3),

\[
\|T(bx) - T(a_m x)\| \leq \|T(bx) - T(bx)\| + \|T(a_m x) - bT(x)\| \to 0 \quad \text{as} \quad m \to \infty \tag{3.4}
\]

for all \( x \in A B \). By (3.1) and (3.4), \( T(ax) = aT(x) \) for all \( a \in A_1^+ \cup \{i\} \) and all \( x \in A B \).

The rest of the proof is similar to the proof of Theorem 3.2.

**Theorem 3.4.** Let \( q \) be not an integer. Let \( f : A B \to A C \) be a mapping with \( f(0) = 0 \) such that

\[
\|D_a f(x_1, \ldots, x_n)\| \leq \varphi(x_1, \ldots, x_n)
\]

for all \( a \in A_1^+ \cup \{i\} \) and all \( x_1, \ldots, x_n \in A B \). If \( f(\lambda x) \) is continuous in \( \lambda \in \mathbb{R} \) for each fixed \( x \in A B \), then there exists a unique \( A \)-linear mapping \( T : A B \to A C \) satisfying (2.ii).

**Proof.** Put \( a = 1 \in A_1^+ \). By Theorem 3.1, there exists a unique \( \mathbb{R} \)-linear mapping \( T : A B \to A C \) satisfying (2.ii).

The rest of the proof is similar to the proof of Theorem 3.2.

**Theorem 3.5.** Let \( A \) be a unital \( C^* \)-algebra of real rank \( 0 \), and \( q \) not an integer. Let \( f : A B \to A C \) be a mapping with \( f(0) = 0 \) such that

\[
\|D_a f(x_1, \ldots, x_n)\| \leq \varphi(x_1, \ldots, x_n)
\]

for all \( a \in (A_1^+ \cap A_{in}) \cup \{i\} \) and all \( x_1, \ldots, x_n \in A B \). Assume that \( f(ax) \) is continuous in \( a \in A_1 \cup \mathbb{R} \) for each fixed \( x \in A B \), and that the sequence \( \{q^{-i} f(q^j x)\} \) converges uniformly on \( A_1 \) for each fixed \( x \in A B \). Then there exists a unique \( A \)-linear mapping \( T : A B \to A C \) satisfying (2.ii).
Proof. By the same reasoning as the proof of Theorem 3.2, there exists a unique $\mathbb{R}$–linear mapping $T : \mathcal{A}\mathcal{B} \to \mathcal{A}\mathcal{C}$ satisfying (2.ii), and
\[
T(ax) = aT(x)
\]
for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x \in \mathcal{A}\mathcal{B}$. By the continuity of $f$ and the uniform convergence, one can show that $T(ax)$ is continuous in $a \in A_1$ for each $x \in \mathcal{A}\mathcal{B}$.

The rest of the proof is similar to the proof of Theorem 3.3. □

Theorem 3.6. Let $f : \mathcal{A}\mathcal{B} \to \mathcal{A}\mathcal{C}$ be a mapping with $f(0) = 0$ such that
\[
\|D_a f(x_1, \cdots, x_n)\| \leq \varphi(x_1, \cdots, x_n)
\]
for all $a \in A_1^+ \cup \{i\} \cup \mathbb{R}$ and all $x_1, \cdots, x_n \in \mathcal{A}\mathcal{B}$. Then there exists a unique $\mathcal{A}$–linear mapping $T : \mathcal{A}\mathcal{B} \to \mathcal{A}\mathcal{C}$ satisfying (2.ii).

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique additive mapping $T : \mathcal{A}\mathcal{B} \to \mathcal{A}\mathcal{C}$ satisfying (2.ii), and
\[
T(ax) = aT(x)
\]
for all $a \in A_1^+ \cup \{i\} \cup \mathbb{R}$ and all $x \in \mathcal{A}\mathcal{B}$. So the mapping $T : \mathcal{A}\mathcal{B} \to \mathcal{A}\mathcal{C}$ is $\mathbb{R}$–linear, and satisfies
\[
T(ax) = aT(x)
\]
for all $a \in A_1^+ \cup \{i\}$ and all $x \in \mathcal{A}\mathcal{B}$.

The rest of the proof is the same as the proof of Theorem 3.2. □

Similarly, for the case that $\varphi : \mathcal{A}\mathcal{B}^n \to [0, \infty)$ is a function such that
\[
\sum_{j=0}^{\infty} q^j \varphi(q^{-j}x_1, \cdots, q^{-j}x_n) < \infty
\]
for all $x_1, \cdots, x_n \in \mathcal{A}\mathcal{B}$, one can obtain similar results to the theorems given above.


In this section, let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras with norms $\| \cdot \|$ and $\| \cdot \|$, respectively.


We prove the Hyers–Ulam–Rassias stability of algebra homomorphisms between Banach algebras associated with the functional equation (A).

Theorem 4.1. Let $\mathcal{A}$ and $\mathcal{B}$ be real Banach algebras, and $q$ not an integer. Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping with $f(0) = 0$ for which there exists a function $\psi : A \times A \to [0, \infty)$ such that
\[
\tilde{\psi}(x, y) := \sum_{j=0}^{\infty} q^{-j} \psi(q^j x, y) < \infty, \quad (4.1)
\]
\[
\|D_1 f(x_1, \cdots, x_n)\| \leq \varphi(x_1, \cdots, x_n), \quad (4.2)
\]
\[
\|f(x \cdot y) - f(x)f(y)\| \leq \psi(x, y) \quad (4.3)
\]
for all \( x, y, x_1, \cdots, x_n \in A \). If \( f(\lambda x) \) is continuous in \( \lambda \in \mathbb{R} \) for each fixed \( x \in A \), then there exists a unique algebra homomorphism \( T : A \to B \) satisfying (2.ii). Further, if \( A \) and \( B \) are unital, then \( f \) itself is an algebra homomorphism.

**Proof.** Under the assumption (2.i) and (4.ii), in Theorem 3.1, we showed that there exists a unique \( \mathbb{R} \)-linear mapping \( T : A \to B \) satisfying (2.ii). The \( \mathbb{R} \)-linear mapping \( T : A \to B \) was given by

\[
T(x) = \lim_{j \to \infty} q^{-j} f(q^j x)
\]

for all \( x \in A \). Let

\[
R(x, y) = f(x \cdot y) - f(x)f(y)
\]

for all \( x, y \in A \). By (4.ii), we get

\[
\lim_{j \to \infty} q^{-j} R(q^j x, y) = 0
\]

for all \( x, y \in A \). So

\[
T(x \cdot y) = \lim_{j \to \infty} q^{-j} f(q^j (x \cdot y)) = \lim_{j \to \infty} q^{-j} f(q^j x \cdot y)
\]

for all \( x, y \in A \). Hence

\[
T(x)q^{-j} f(q^j y) = T(x) f(y)
\]

for all \( x, y \in A \). Taking the limit in (4.2) as \( j \to \infty \), we obtain

\[
T(xy) = T(x)f(y)
\]

for all \( x, y \in A \). Therefore,

\[
T(x \cdot y) = T(x)f(y)
\]

for all \( x, y \in A \). So \( T : A \to B \) is an algebra homomorphism.

Now assume that \( A \) and \( B \) are unital. By (4.2),

\[
T(y) = T(1 \cdot y) = T(1)f(y) = f(y)
\]

for all \( y \in A \). So \( f : A \to B \) is an algebra homomorphism, as desired. \( \Box \)

**Theorem 4.2.** Let \( A \) and \( B \) be complex Banach algebras. Let \( f : A \to B \) be a mapping with \( f(0) = 0 \) for which there exists a function \( \psi : A \times A \to [0, \infty) \) satisfying (4.i) and (4.iii) such that

\[
\|D(\lambda f)(x_1, \cdots, x_n)\| \leq \varphi(x_1, \cdots, x_n)
\]

for all \( \lambda \in \mathbb{T} \) and all \( x_1, \cdots, x_n \in A \). Then there exists a unique algebra homomorphism \( T : A \to B \) satisfying (2.ii). Further, if \( A \) and \( B \) are unital, then \( f \) itself is an algebra homomorphism.
Proof. Under the assumption \((2.i)\) and \((4.iv)\), in Corollary 2.2, we showed that there exists a unique \(\mathbb{C}\)-linear mapping \(T : A \rightarrow B\) satisfying \((2.ii)\).

The rest of the proof is the same as the proof of Theorem 4.2. \(\square\)

**Theorem 4.3.** Let \(A\) and \(B\) be complex Banach \(*\)-algebras. Let \(f : A \rightarrow B\) be a mapping with \(f(0) = 0\) for which there exists a function \(\psi : A \times A \rightarrow [0, \infty)\) satisfying \((4.i)\) and \((4.iii)\) such that

\[
\|D\lambda f(x_1, \ldots, x_n)\| \leq \varphi(x_1, \ldots, x_n) \quad (4.iv)
\]

\[
\|f(x^*) - f(x)^*\| \leq \varphi(x, \ldots, x) \quad (4.v)
\]

for all \(\lambda \in \mathbb{T}^1\) and all \(x, x_1, \ldots, x_n \in A\). Then there exists a unique \(*\)-algebra homomorphism \(T : A \rightarrow B\) satisfying \((2.ii)\). Further, if \(A\) and \(B\) are unital, then \(f\) itself is a \(*\)-algebra homomorphism.

Proof. By the same reasoning as the proof of Theorem 4.2, there exists a unique \(\mathbb{C}\)-linear mapping \(T : A \rightarrow B\) satisfying \((2.ii)\).

Now

\[
q^{-j}\|f(q^j x^*) - f(q^j x)^*\| \leq q^{-j}\varphi(q^j x, \ldots, q^j x)
\]

for all \(x \in A\). Thus

\[
q^{-j}\|f(q^j x^*) - f(q^j x)^*\| \rightarrow 0
\]

as \(n \rightarrow \infty\) for all \(x \in A\). Hence

\[
T(x^*) = \lim_{j \rightarrow \infty} q^{-j}f(q^j x^*) = \lim_{j \rightarrow \infty} q^{-j}f(q^j x)^* = T(x)^*
\]

for all \(x \in A\).

The rest of the proof is the same as the proof of Theorem 4.2. \(\square\)

Similarly, for the case that \(\varphi : A^n \rightarrow [0, \infty)\) and \(\psi : A \times A \rightarrow [0, \infty)\) are functions such that

\[
\sum_{j=0}^{\infty} q^j \varphi(q^{-j} x_1, \ldots, q^{-j} x_n) < \infty,
\]

\[
\sum_{j=0}^{\infty} q^j \psi(q^{-j} x, y) < \infty
\]

for all \(x, y, x_1, \ldots, x_n \in A\), one can obtain similar results to the theorems given above.

**References**


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