A NOTE ON DEVELOPABILITY AND METRIZABILITY

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Abstract. In this paper, we answer two questions of P. Fletcher and W. Lindgren [1] and R. Gittings [4], one of which is partially answered. We prove that a space $X$ is developable if and only if it is $w\Delta$-space with a quasi-$G_\delta^*$-diagonal; a space $X$ is developable if and only if it is quasi-developable, $\beta$-space; a space $X$ is developable if and only if it is $\beta$, quasi-$\gamma$-space with a quasi-$G_\delta^*$-diagonal; a space is metrizable if and only if it is $wM$-space with a quasi-$G_\delta^*$-diagonal.

1. Introduction

In this brief note we present some conditions which imply developability and metrizability, and consequently we give a full positive answer to Fletcher and Lindgren’s question [1] and a partial answer to R. Gittings’s question [4] respectively: is every quasi-developable $\beta$-space developable? Is every $wM$-space with $G_\delta$-diagonal metrizable?

In [13], the author has obtained a factorization of quasi-developability into two parts: a space $X$ is quasi developable if and only if it is a quasi-$w\Delta$-space with a quasi-$G_\delta^*$-diagonal. This result plays an important role in obtaining the results in this paper.

A COC-map (= countable open covering map) for a topological space $X$ is a function $g$ from $N \times X$ into the topology of $X$ such that for every $x \in X$ and $n \in N$, $x \in g(n, x)$ and $g(n + 1, x) \subseteq g(n, x)$. It is well known that many important classes of generalized metrizable spaces can be characterized in terms of a COC-map. In particular, $X$ is developable [5] ($w\Delta$-space) if and only if $X$ has a COC-map $g$ such that if $\{p, x_n\} \subseteq g(n, y_n)$ for all $n$, then $(x_n)$ converges to $p$ (then $(x_n)$ has a cluster point).

A space $X$ is called quasi-$\gamma$ [10] if and only if $X$ has a COC-map $g$ such that if $x_n \in g(n, y_n)$ for each $n \in N$, and the sequence $(y_n)$ converges in $X$, then the sequence $(x_n)$ has a cluster point; a space $X$ is called semi-stratifiable [7] ($\beta$-space [6]) if and only if $X$ has a COC-map $g$ such that if $x \in g(n, x_n)$ for each $n \in N$ then $x$ is a cluster point of $(x_n)$ ($(x_n)$ has a cluster point).

Let $\mathcal{G} = \{G_n\}_{n \in N}$ be a sequence of open families of $X$. For each $x \in X$, let $c(x) = c_\mathcal{G}(x) = \{n : x \in G_n^* = \bigcup \{G : G \in G_n\}\}$. A space $X$ has a quasi-$G_\delta^*$-diagonal [13] (quasi-$G_\delta^*(2)$-diagonal) if there is such a sequence $\mathcal{G}$ such that for any distinct $x, y \in X$, there exists $n \in N$ such that $x \in st(x, G_n) \subset X - \{y\}$.

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(x \in \text{st}^2(x, G_n) \subset X - \{y\}); a space X is called a quasi-\(w\Delta\)-space [13] if X has such a sequence \(G\) such that

1. for all \(x\), \(c(x)\) is infinite.
2. if \(\langle x_n \rangle\) is a sequence with \(x_n \in \text{st}(x, G_n)\) for all \(n \in c(x)\) then \(\langle x_n \rangle\) has a cluster point.

If we take \(G\) as a sequence of open covers of \(X\) with the condition (2) \((\langle x_n \rangle)\) is a sequence with \(x_n \in \text{st}^2(x, G_n)\) for all \(n \in \mathbb{N}\) then \(\langle x_n \rangle\) has a cluster point, then \(X\) is a \(w\Delta\)-space (\(wM\)-space).

A space \(X\) is called \(c\)-semi-stratifiable [10] if for each \(x \in X\), there is a sequence \(\langle g(n, x) \rangle\) of open neighborhoods of \(x\) such that for each compact set \(K \subset X\), if \(g(n, K) = \bigcup\{g(n, x) : x \in K\}\), then \(\cap\{g(n, K) : n \geq 1\} = K\). The \(\text{COC}\)-map \(g : \mathbb{N} \times X \to \tau\) is called a \(c\)-semi-stratification of \(X\).

A space \(X\) is quasi-developable if there exists a sequence \(\langle Q_n \rangle\) of families of open subsets of \(X\) such that for each \(x \in X\), \(\{\text{st}(x, Q_n) : n \in \mathbb{N}\} - \{\emptyset\}\) is a base at \(x\).

All spaces will be assumed regular, unless stated otherwise.

2. Main Results

Lemma 2.1. Let \(X\) be a space with a quasi-\(G_\delta^*\)-diagonal sequence. Then \(X\) has a quasi-\(G_\delta^*\)-diagonal sequence \(\langle G_n : n \in \mathbb{N}\rangle\) such that for each \(x \in X\) there is an infinite subset \(d(x) \subset c_G(x)\) such that if \(x_n \in \text{st}(x, G_n)\) for each \(n \in d(x)\) then \(\langle x_n \rangle\) either clusters at \(x\) or it does not cluster at all.

Proof. Let \(\langle H_n : n \in \mathbb{N}\rangle\) be a quasi-\(G_\delta^*\)-diagonal sequence of \(X\). Without loss of generality we may assume that \(c_H(x)\) is infinite for each \(x \in X\) and \(H_1 = \{X\}\). Let \(F\) denote the collection of non-empty finite subsets of \(\mathbb{N}\). For each \(F \in \mathcal{F}\) let

\[G_F = \{\bigcap_{i \in F} H_i : H_i \in \mathcal{H}_i\}.\]

For \(n \in \mathbb{N}\) and \(x \in X\), set \(F_n(x) = c_H(x) \cap \{1, 2, \ldots, n\}\). Put \(d(x) = \{F_n(x) : n \in \mathbb{N}\}\). Note that \(d(x) \subset c_G(x)\). Since \(c_H(x)\) is infinite, \(d(x)\) is infinite. Because \(F_n(x) \subset F_m(x)\) for \(m \geq n\), \(\text{st}(x, G_{F_m(x)}) \subset \text{st}(x, G_{F_n(x)})\) for \(m \geq n\).

For each \(n \in \mathbb{N}\) suppose that \(x_n \in \text{st}(x, G_{F_n(x)})\). Then for \(m \geq n\) we have

\[
\{x_m \mid m \geq n\} \subset \text{st}(x, G_{F_n(x)}).
\]

Since \(\cap_{n \in \mathbb{N}} \text{st}(x, G_{F_n(x)}) = \{x\}\) it follows that either \(\langle x_n \rangle\) clusters at \(x\) or does not cluster at all. \(\square\)

Remark 2.2. Let \(X\) be a space and \(\langle G_n : n \in \mathbb{N}\rangle\) a countable family of collections of open subsets of a space \(X\), such that for all \(x\), \(c(x) = \{n \in \mathbb{N} : x \in G_n^*\}\) is infinite. Consider the following condition on \(\langle G_n : n \in \mathbb{N}\rangle\): if \(\langle x_n : n \in \mathbb{N}\rangle\) is a sequence with \(x_n \in \text{st}(x, G_n)\) for all \(n \in c(x)\) then \(x\) is a cluster point of \(\langle x_n : n \in \mathbb{N}\rangle\).

For all spaces, this condition is equivalent to the following condition: for each point \(x \in X\) the set \(\text{st}(x, G_n)\) is nonempty for infinitely many \(n\) and the nonempty sets of the form \(\text{st}(x, G_n)\) form a local base at \(x\) for all \(x \in X\). Thus the condition above is a characterization of a quasi-developable space.

Theorem 2.3. Every quasi-developable space is a \(c\)-semi-stratifiable space.
Proof. Let \( \langle G_n : n \in \mathbb{N} \rangle \) be a quasi-development sequence in a space \( X \). Define
\[
g(n, x) = \begin{cases} 
st(x, G_n) & \text{if } x \in G_n^*, \\ X & \text{if } x \notin G_n^*. \end{cases}
\]

Let \( h(n, x) = \cap_{i=1}^n g(i, x) \). We prove that \( h(n, x) \) is a \( c\)-semi-stratifiable-map. We claim that \( C = \cap_{n \in \mathbb{N}} h(n, C) \) for any compact \( C \subseteq X \), where \( h(n, C) = \bigcup_{c \in C} h(n, c) \). As \( G_1 = \{X\} \) it follows readily that \( C \subseteq h(n, C) \) so we need only prove the reverse inclusion. For this, let \( y \in \cap h(n, C) \), so \( y \in h(n, c_n) \) for some \( c_n \in C \). Then \( y \in st(c_n, G_n) \) for infinitely many \( n \in \mathbb{N} \). It follows that \( c_n \in st(y, G_n) \) for infinitely many \( n \in \mathbb{N} \). From Remark 2.2, \( \langle c_n \rangle \) clusters at \( y \). Hence, \( y \in C \). □

The following lemma is proved by Martin in [11].

**Lemma 2.4.** A space is semi-stratifiable if and only if it is a \( c \)-semi-stratifiable \( \beta \)-space.

**Proof.** The only if part is clear. Conversely, let \( X \) be a regular \( c \)-semi-stratifiable \( \beta \)-space. Let \( f \) be a \( c \)-semi-stratifiable-map and \( g \) be a \( \beta \)-map. Define \( h(n, x) = f(n, x) \cap g(n, x) \). It is clear that \( h \) is a \( c \)-semi-stratifiable, \( \beta \)-map. Since \( X \) is a regular and \( h \) is a \( c \)-semi-stratifiable, \( \beta \)-map, \( h(n + 1, x) \subseteq h(n, x) \) for all \( x \in X \) and all \( n \in \mathbb{N} \). Moreover, if \( x \in h(n, x_n) \) for \( n \in \mathbb{N} \), then the sequence \( \langle x_n \rangle \) has a cluster point. Now to prove that \( h \) is a semi-stratifiable-map, let \( x \in h(n, x_n) \) for \( n \in \mathbb{N} \). We must prove that the sequence \( \langle x_n \rangle \) is convergent to \( x \).

The sequence \( \langle x_n \rangle \) has at least one cluster point. Moreover, it is easy to show that every subsequence of \( \langle x_n \rangle \) also has at least one cluster point. Suppose \( p \) is a cluster point of \( \langle x_n \rangle \) and that \( p \neq x \). Choose a subsequence of \( \langle x_{n_i} \rangle \) of \( \langle x_n \rangle \) such that \( x_{n_i} \in g(i, p) \) for \( i \in \mathbb{N} \) and \( x \neq x_{n_i} \) for all \( i \). Since every subsequence of \( \langle x_{n_i} \rangle \) has a cluster point, it follows that \( \langle x_{n_i} \rangle \) converges to \( p \). Therefore \( K = \{p\} \cup \{x_{n_i}\} \) is compact. There exists \( m \in \mathbb{N} \) such that \( x \notin h(m, K) \). Choose \( k > m \) such that \( x_k \in K \); then \( x \notin h(m, x_k) \). But \( h(k, x_k) \subseteq h(m, x_k) \), so \( x \notin h(k, x_k) \), which is a contradiction. It follows that \( x \) is the only cluster point of \( \langle x_n \rangle \). Since every subsequence of \( \langle x_n \rangle \) has a cluster point, necessarily \( \langle x_n \rangle \) converges to \( x \). □

**Theorem 2.5.** A space is developable if and only if it is a quasi-developable \( \beta \)-space.

**Proof.** The only if part is clear. The converse follows from Lemma 2.4 and Theorem 2.3.

**Corollary 2.6.** A space \( X \) is developable if and only if \( X \) is a \( w\Delta \)-space with a quasi-\( G^*_\delta \)-diagonal.

**Proof.** This follows from [13, Theorem 3.1] since every \( w\Delta \)-space is \( \beta \)-space. □

**Theorem 2.7.** A space \( X \) is developable if and only if it is \( \beta \), quasi-\( \gamma \)-space with a quasi-\( G^*_\delta \)-diagonal.

**Proof.** The necessity of the conditions is obvious. Conversely, let \( f \) be a \( \beta \)-map and \( g \) a quasi-\( \gamma \)-map of \( X \). Define \( h(n, x) = f(n, x) \cap g(n, x) \). It is clear that \( h \) is a \( \beta \) and quasi-\( \gamma \)-map of \( X \). We prove that \( h \) is a \( w\Delta \)-map of \( X \). Let \( \{x, x_n\} \subset h(n, y_n) \). By the \( \beta \)-condition, \( \langle y_n \rangle \) converges and so by the quasi-\( \gamma \)-condition, \( \langle x_n \rangle \) has a
cluster point. Thus $h$ is $w\Delta$-map of $X$. From Corollary 2.6, it follows that $X$ is a developable space.

**Corollary 2.8.** A space is metrizable if and only if it is a $wM$-space with a quasi-$G^*_\delta$-diagonal.

**Proof.** Let $X$ be a regular, $wM$-space with a quasi-$G^*_\delta$-diagonal. Every $wM$-space is a $w\Delta$-space so that (by Corollary 2.6) $X$ is developable. Every developable, $wM$-space is metrizable, hence this completes the proof.

**Question 1.** Is every quasi-$w\Delta$-space (quasi-$wM$-space) with $G^*_\delta$-diagonal necessarily developable (metrizable)?

We answer this question negatively.

**Example 2.9.** There is a $p$-adic analytic manifold which is separable, submetrizable, quasi-$wM$, quasi-developable, but not perfect ([12, Example 7.4.7]). This example can also serve as a quasi-semi-stratifiable space (see [8] for the definition) which has a $G^*_\delta$-diagonal but which is not semi-stratifiable.

**Example 2.10.** There is a quasi-developable manifold which has a $G_\delta$-diagonal but not a $G^*_\delta$-diagonal (see [3, Example 2.2]) This example can also serve as a quasi-$w\Delta$ manifold which is not $w\Delta$. (It is not even a $\beta$-manifold).

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**References**


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