A WIENER–HOPF DYNAMICAL SYSTEM FOR VARIATIONAL INEQUALITIES

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(Received February 2001)

Abstract. In this paper, we use the Wiener–Hopf technique to suggest and analyze a dynamical system for variational inequalities. We prove that the globally asymptotic stability of this dynamical system requires only the pseudomonotonicity of the operator, which is a weaker condition than monotonicity.

1. Introduction

Variational inequalities are being used as a mathematical programming tool to study equilibrium problems arising in economics, transportation, elasticity and structural analysis; see, for example, [1–19]. Using the projection technique, one can establish the equivalence between variational inequalities and fixed-point problems. This alternative equivalence has been used to suggest and study two types of projected dynamical systems associated with variational inequalities. The first type, which is due to Friesz et al [3], is designated as global projected dynamical systems. The second type, due to Dupuis and Nagurney [2], is designate as local-projected dynamical systems. The novel and innovative feature of the projected dynamical systems is that the set of the stationary points of the dynamical systems correspond to the set of the solutions of the variational inequalities. Consequently, equilibrium problems which can be formulated in the setting of the variational inequalities now can be studied in the more general setting of dynamical systems. This approach has resulted numerous dynamical models in a variety of fields along with algorithms and computational results; see, for example [1–4, 15–19].

Equally important is the field of the Wiener–Hopf equations, which was introduced and studied by Shi [15] and Robinson [14] independently. Using the projection technique, one can prove that variational inequalities are equivalent to Wiener–Hopf equations. This equivalence has been used to develop some efficient and powerful numerical methods for solving variational inequalities, see [9–11]. The theory of the Wiener–Hopf equations is very general and flexible. This theory can be used to interpret the basic principles of mathematical and physical sciences in the form of simplicity and elegance. For recent applications, numerical methods, sensitivity analysis and generalizations of Wiener–Hopf equations, see [8–11] and the references therein.

Inspired and motivated by the research going in these fields, we use the Wiener–Hopf equations technique to suggest and analyze a new dynamical system. We prove that the Wiener–Hopf dynamical system has the globally asymptotic stability

2000 AMS Mathematics Subject Classification: 49J40, 90C33.
Key words and phrases: Variational inequalities, projected dynamical systems, global convergence, stability.
property for the pseudomonotone operator. It is known that the monotonicity implies pseudomonotonicity, but the converse is not true. This shows that the result regarding the convergence of the projected dynamical system obtained in this paper represents a refinement of the previously known results.

2. Formulations and Basic Facts

Let $\mathbb{R}^n$ be a Euclidean space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. Let $K$ be a closed convex set in $\mathbb{R}^n$ and let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a nonlinear operator. We now consider the problem of finding $u \in K$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \text{for all } v \in K. \quad (2.1)$$

Problem (2.1) is called the variational inequality, which was introduced and studied by Stampacchia [16] in 1964. It has been shown that a large class of obstacle, unilateral, contact, free, moving, and equilibrium problems arising in regional, physical, mathematical, engineering, economics and applied sciences can be studied in the unified and general framework of the variational inequalities (2.1), see [1–19] and the references therein.

We also need the following well known results and concepts.

**Definition 2.1.** For all $u, v \in \mathbb{R}^n$, an operator $A : \mathbb{R}^n \to \mathbb{R}^n$ is said to be

(a) **monotone** if

$$\langle Au - Av, u - v \rangle \geq 0$$

(b) **pseudomonotone** if

$$\langle Au, v - u \rangle \geq 0 \quad \text{implies} \quad \langle Av, v - u \rangle \geq 0$$

(c) **Lipschitz continuous** if there exists a constant $\beta > 0$ such that

$$\| Au - Av \| \leq \beta \| u - v \|.$$

It is well known that monotonicity implies pseudomonotonicity, but not conversely, see [5].

**Lemma 2.2 ([6]).** For a given $z \in \mathbb{R}^n$, $u \in K$ satisfies the inequality

$$\langle u - z, v - u \rangle \geq 0, \quad \text{for all } v \in K, \quad (2.2)$$

if and only if $u = P_K z$, where $P_K$ is the projection of $\mathbb{R}^n$ onto the convex set $K$. Furthermore, the operator $P_K$ is nonexpansive.

This result plays an important role in obtaining the main results of this paper. Invoking Lemma 2.2, one can easily show that the problem (2.1) is equivalent to the fixed-point problem.

**Lemma 2.3 ([6]).** The function $u \in K$ is a solution of problem (2.1) if and only if $u \in K$ satisfies the relation

$$u = P_K [u - \rho Au],$$

where $\rho > 0$ is a constant.
We now define the residue vector \( R(u) \) by the relation
\[
R(u) = u - P_K[u - \rho Au].
\] (2.3)
Invoking Lemma 2.4, one can easily conclude that \( u \in K \) is a solution of problem (2.1) if and only if \( u \in K \) is a zero of the equation
\[
R(u) = 0.
\] (2.4)

We now consider the Wiener-Hopf equations. To be more precise, let \( Q_K = I - P_K \), where \( I \) is the identity operator and \( P_K \) is the projection of \( \mathbb{R}^n \) onto \( K \). For a given nonlinear operator \( A : \mathbb{R}^n \rightarrow \mathbb{R}^n \), we consider the problem of finding \( z \in \mathbb{R}^n \) such that
\[
\rho A P_K z + Q_K z = 0,
\] (2.5)
where \( \rho > 0 \) is a constant. Equations of the type (2.5) are called the Wiener-Hopf (normal) equations, which were introduced and studied by Shi [15] and Robinson [14] independently. Using the projection technique, one can establish the equivalence between the Wiener-Hopf equations and the variational inequalities. This interplay between these problems is very useful and has been successfully applied to suggest and analyze some efficient numerical methods for solving variational inequalities and related optimization problems. For the applications and numerical methods of the Wiener-Hopf equations, see [8-10, 13-15] and the references therein. We need the following well known result, which shows that the Wiener-Hopf equations are equivalent to the fixed-point problem.

**Lemma 2.4 ([8], [14], [15]).** The variational inequalities (2.1) have a solution \( u \in K \) if and only if the Wiener-Hopf equations (2.5) have a solution \( z \in \mathbb{R}^n \), where
\[
u = P_K z \quad (2.6)
\]
\[
z = u - \rho Au.
\] (2.7)
Using (2.6) and (2.7), the Wiener-Hopf equations (2.5) can be written as
\[
u - \rho Au - P_K[u - \rho Au] + \rho A P_K[u - \rho Au] = 0.
\] (2.8)
Thus it is clear from Lemma 2.4 that \( u \in K \) is a solution of the variational inequality (2.1) if and only if \( u \in K \) satisfies the equation (2.8).

Using this equivalence, we suggest a new dynamical system associated with variational inequality (2.1) as
\[
\frac{du}{dt} = \lambda[P_K[u - \rho Au] - \rho A P_K[u - \rho Au] + \rho Au - u],
\]
\[
= \lambda[-R(u) + \rho Au - \rho A P_K[u - \rho Au]], \quad u(t_0) = u_0 \in K,
\] (2.9)
where \( \lambda \) is a constant. Here the right-hand side is associated with projection and hence is discontinuous on the boundary of \( K \). It is clear from the definition that the solution to (2.9) belongs to the constraint set \( K \). This implies that the results such as the existence, uniqueness and continuous dependence on the given data can be studied. It is worth mentioning that the Wiener-Hopf dynamical system (2.9) is different from the one considered and studied in [1-4, 17 - 19]. In passing, we remark that the Wiener-Hopf equations technique has been used to suggest some efficient numerical methods for solving unilateral and obstacle problems. We
expect that the Wiener–Hopf dynamical system will be useful in analyzing the stability and qualitative properties of complicated problems arising in economics and transportation.

**Definition 2.5.** The dynamical system is said to converge to the solution set $K^*$ of (2.1) if and only if, irrespective of the initial point, the trajectory of the dynamical system satisfies

$$\lim_{t \to \infty} \text{dist}(u(t), K^*) = 0,$$  
(2.10)

where

$$\text{dist}(u, K^*) = \inf_{v \in K^*} \|u - v\|.$$  

It is easy to see that, if the set $K^*$ has a unique point $u^*$, then (2.10) implies that

$$\lim_{t \to \infty} u(t) = u^*.$$  

If the dynamical system is still stable at $u^*$ in the Lyapunov sense, then the dynamical system is globally asymptotically stable at $u^*$.

**Definition 2.6.** The dynamical system is said to be globally exponentially stable with degree $\eta$ at $u^*$ if and only if, irrespective of the initial point, the trajectory of the system $u(t)$ satisfies

$$\|u(t) - u^*\| \leq \mu_1 \|u(t_0) - u^*\| \exp(-\eta(t - t_0)), \quad \text{for all } t \geq t_0,$$

where $\mu_1$ and $\eta$ are positive constants independent of the initial point. It is clear that globally exponential stability is necessarily globally asymptotical stability and the dynamical system converges arbitrarily fast.

From now onward we assume that $K^*$ is nonempty and is bounded, unless otherwise specified.

### 3. Main Results

In this section we study the main properties of the dynamical systems and analyze the global stability of the systems. First of all, we discuss the existence and uniqueness of the dynamical system (2.9) and this is the main motivation of our next result.

**Theorem 3.1.** Let the operator $A$ be a Lipschitz continuous operator. Then, for each $u_0 \in \mathbb{R}^n$, there exists a unique continuous solution $u(t)$ of dynamical system (2.9) with $u(t_0) = u_0$ over $[t_0, \infty]$.

**Proof.** Let

$$G(u) = \lambda\{P_K[u - \rho Au] - \rho AP_K[u - \rho Au] + \rho Au - u\},$$

where $\lambda > 0$ is a constant. For all $u, v \in \mathbb{R}^n$, we have

$$\|G(u) - G(v)\|$$

$$\leq \lambda\{|P_K[u - \rho Au] - P_K[v - \rho Av] - \rho AP_K[u - \rho Au] + \rho AP_K[v - \rho Av]| + \rho\|Au - Av\| + \|u - v\|\}$$

$$\leq \lambda\{(1 + \rho\beta)\|u - v\| + (1 + \rho\beta)\|(u - v) - \rho(Au - Av)\|\}$$

$$\leq \lambda(1 + \rho\beta)(2 + \rho\beta)\|u - v\|,$$
where $\beta > 0$ is a Lipschitz constant of the operator $A$. This implies that the operator $G(u)$ is a Lipschitz continuous in $\mathbb{R}^n$. So, for each $u_0 \in \mathbb{R}^n$, there exists a unique and continuous solution $u(t)$ of the dynamical system of (2.9), defined in a interval $t_0 \leq t < T$ with the initial condition $u(t_0) = u_0$. Let $[t_0, T)$ be its maximal interval of existence; we show that $T = \infty$. Consider

$$
\|G(u)\| = \lambda \|P_K[u - \rho Au] - \rho [AP_K[u - \rho Au] + \rho Au - u]\|
\leq \lambda \|P_K[u - \rho Au] - u\| + \lambda \rho \beta \|P_K[u - \rho Au] - u\|
= (1 + \lambda \rho \beta) \|P_K[u - \rho Au] - u\|
\leq (1 + \lambda \rho \beta) \{\|P_K[u - \rho Au] - P_K[u]\| + \rho \|P_K[u] - P_K[u^*]\| + \|P_K[u^*] - u\|\}
\leq (1 + \lambda \rho \beta) \{\rho \|Au\| + \|u - u^*\| + \|P_K[u^*] - u\|\}
\leq (1 + \lambda \rho \beta) \{(2 + \rho \beta) \|u\| + \|P_K[u^*]\| + \|u^*\|\}
= (1 + \lambda \rho \beta)(2 + \rho \beta) \|u\| + \lambda(1 + \rho \beta) \{\|P_K[u^*]\| + \|u^*\|\},
$$
for any $u \in \mathbb{R}^n$, then

$$
\|u(t)\| \leq \|u_0\| + \int_{t_0}^{t} \|Tu(s)\|ds
\leq (\|u_0\| + k_1(t - t_0)) + k_2 \int_{t_0}^{t} \|u(s)\|ds,
$$
where $k_1 = \lambda(1 + \rho \beta) \{\|u^*\| + \|P_K[u^*]\|\}$ and $k_2 = \lambda(2 + \rho \beta)(1 + \rho \beta)$. Hence, by invoking Gronwall Lemma [4], we have

$$
\|u(t)\| \leq \{\|u_0\| + k_1(t - t_0)\}e^{k_2(t - t_0)}, \quad t \in [t_0, T].
$$
This shows that the solution $u(t)$ is bounded on $[t_0, T)$. So $T = \infty$. □

We now study the stability of the Wiener–Hopf dynamical system (2.9). The analysis is in the spirit of Xia and Wang [16, 17] and Noor [11].

**Theorem 3.2.** Let $A$ be a pseudomonotone Lipschitz continuous operator. Then the Wiener–Hopf dynamical system (2.9) is stable in the sense of Lyapunov and globally converges to the solution subset of (2.1).

**Proof.** Since the operator $A$ is a Lipschitz continuous operator, it follows from Theorem 3.1, that the Wiener–Hopf dynamical system (2.9) has a unique continuous solution $u(t)$ over $[t_0, T)$ for any fixed $u_0 \in K$. Let $u(t) = u(t, t_0; u_0)$ be the solution of the initial value problem (2.9). For a given $u^* \in K$, consider the following Lyapunov function

$$
L(u) = \|u - u^*\|^2, \quad u \in \mathbb{R}^n.
$$
(3.1)

It is clear that $\lim_{n \to \infty} L(u_n) = +\infty$ when ever the sequence $\{u_n\} \subset K$ and $\lim_{n \to \infty} u_n = +\infty$. Consequently, we conclude that the level sets of $L$ are bounded. Let $u^* \in K$ be a solution of (2.1). Then

$$
\langle Au^*, v - u^* \rangle \geq 0, \quad \forall v \in K,
$$
which implies that
\[ \langle Av, v - u^* \rangle \geq 0, \] (3.2)
since the operator \( A \) is pseudomonotone.

Taking \( v = P_K[u - \rho Au] \) in (3.2), we have
\[ \langle AP_K[u - \rho Au], P_K[u - \rho Au] - u^* \rangle \geq 0. \] (3.3)

Setting \( v = u^* \), \( u = P_K[u - \rho Au] \), and \( z = u - \rho Au \) in (2.2), we have
\[ \langle P_K[u - \rho Au] - u + \rho Au, u^* - P_K[u - \rho Au] \rangle \geq 0. \] (3.4)

Adding (3.3), (3.4) and using (2.3), we obtain
\[ \langle -R(u) + \rho Au - \rho AP_K[u - \rho Au], u^* - u + R(u) \rangle \geq 0, \]
which implies that
\[ \langle u - u^*, R(u) - \rho Au + \rho AP_K[u - \rho Au] \rangle \geq \|R(u)\|^2 \]
\[ - \rho \langle R(u), Au - AP_K[u - \rho Au] \rangle \geq (1 - \delta \rho)\|R(u)\|^2, \] (3.5)

where we have used the fact that the operator \( A \) is Lipschitz continuous with constant \( \delta > 0 \). Thus, from (2.9), (3.1) and (3.5), we have
\[
\frac{d}{dt} L(u) = \frac{dL}{du} \frac{du}{dt} = 2\lambda \langle u - u^*, -R(u) + \rho Au - \rho AP_K[u - \rho Au] - u \rangle \\
= 2\lambda (1 - \delta \rho) \langle u - u^*, -R(u) \rangle \\
\leq -2\lambda (1 - \delta \rho)\|R(u)\|^2 \leq 0.
\]

This implies that \( L(u) \) is a global Lyapunov function for the system (2.9) and the Wiener–Hopf dynamical system (2.9) is stable in the sense of Lyapunov. Since \( \{u(t) : t \geq t_0\} \subset K_0 \), where \( K_0 = \{u \in K : L(u) \leq L(u_0)\} \) and the function \( L(u) \) is continuously differentiable on the bounded and closed set \( K \), it follows from LaSalle’s invariance principle that the trajectory will converge to \( \Omega \), the largest invariant subset of the following subset:
\[ E = \left\{ u \in K; \frac{dL}{dt} = 0 \right\}. \]

Note that, if \( \frac{dL}{dt} = 0 \), then
\[ \|u - P_K[u - \rho Au]\|^2 = 0, \]
and hence \( u \) is an equilibrium point of the dynamical system (2.9), that is,
\[ \frac{du}{dt} = 0. \]

Conversely, if \( \frac{du}{dt} = 0 \), then it follows that \( \frac{dL}{dt} = 0 \). Thus, we conclude that
\[ E = \left\{ u \in K : \frac{du}{dt} = 0 \right\} = K_0 \cap K^*, \]
which is a nonempty, convex and invariant set contained in the solution set $K^*$. So
\[
\lim_{t \to \infty} \text{dis}(u(t), E) = 0.
\]
Therefore, the dynamical system (2.9) converges globally to the solution set of (2.1). In particular, if the set $E = \{u^*\}$, then
\[
\lim_{t \to \infty} u(t) = u^*.
\]
Hence the system (2.9) is globally asymptotically stable. □

**Theorem 3.3.** Let the operator $T$ be Lipschitz continuous with a constant $\beta > 0$. If $\lambda < 0$, then the Wiener-Hopf dynamical system (2.9) converges globally exponentially to the unique solution of the variational inequality (2.1).

**Proof.** From Theorem 3.1, we see that there exists a unique continuously differentiable solution of the dynamical system (2.9) over $[t_0, \infty)$. Then
\[
\frac{dL}{dt} = 2\lambda(u(t) - u^*, P_K[u(t) - \rho Au(t)] - \rho AP_K[u(t) - \rho Au(t)] + \rho Au(t) - u(t))
\]
\[
= -2\lambda\|u(t) - u^*\|^2 + 2\lambda(u(t) - u^*, P_K[u(t) - \rho Au(t)] - \rho AP_K[u(t) - \rho Au(t)] + \rho Au(t) - u^*),
\]
where $u^* \in K$ is the solution of the variational inequality (2.1). Thus
\[
u^* = P_K[u^* - \rho Au^*] - \rho AP_K[u^* - \rho Au^*] + \rho Au^*.
\]
Now, using the nonexpansivity of $P_K$ and Lipschitz continuity of the operator $A$, we have
\[
\|P_K[u - \rho Au] - \rho AP_K[u - \rho Au] + \rho Au - P_K[u^* - \rho Au^*] + \rho AP_K[u^* - \rho Au^*] - \rho Au^*\|
\]
\[
\leq \|P_K[u - \rho Au] - P_K[u^* - \rho Au^*]\|
\]
\[
+ \rho\|AP_K[u - \rho Au] - AP_K[u^* - \rho Au^*]\| + \rho\|Au - Au^*\|
\]
\[
\leq (1 + \rho\beta)\|P_K[u - \rho Au] - P_K[u^* - \rho Au^*]\| + \rho\|u - u^*\|
\]
\[
\leq (1 + \rho\beta)(\|u - u^*\| + \rho(Au - Au^*)) + \rho\|u - u^*\|
\]
\[
\leq (1 + \rho\beta)^2 + \rho\|u - u^*\|.
\]
From (3.6) and (3.7), we have
\[
\frac{d}{dt}\|u(t) - u^*\|^2 \leq +2\alpha\lambda\|u(t) - u^*\|^2,
\]
where
\[
\alpha = (3 + \rho\beta)\rho\beta.
\]
Thus, for $\lambda = -\lambda_1$, where $\lambda_1$ is a positive constant, we have
\[
\|u(t) - u^*\| \leq \|u(t_0) - u\|e^{-\alpha\lambda_1(t-t_0)},
\]
which shows that the trajectory of the solution of the dynamical system (2.9) will globally exponentially converge to the unique solution of the variational inequality (2.1). □
4. Extensions

We would like to mention that the results obtained in this paper can be extended for variational inclusions (mixed variational inequalities). More precisely, we consider the problem of finding \( u \in \mathbb{R}^n \) such that

\[
0 \in Au + \partial \varphi(u),
\]

(4.1)

where \( \partial \varphi \) is the subdifferential of a proper, convex and lower–semicontinuous function \( \varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\} \). Problem (4.1) is called the variational inclusion or the mixed variational inequality. If \( \varphi \) is the indicator function of a closed convex set \( K \) in \( \mathbb{R}^n \), then problem (4.1) is equivalent to the variational inequality (2.1). Using the resolvent operator technique, it can be shown that problem (4.1) is equivalent to the fixed–point problem of the form:

\[
u = J_\varphi[u - \rho Au],
\]

(4.2)

where \( J_\varphi \equiv (I + \rho \partial \varphi)^{-1} \) is the resolvent operator associated with the maximal monotone operator \( \partial \varphi \) and \( I \) is the identity operator. Related to the variational inclusion (4.1), we consider the problem of finding \( z \in \mathbb{R}^n \) such that

\[
\rho T J_\varphi z + R_\varphi z = 0,
\]

(4.3)

where \( R_\varphi = I - J_\varphi \) and \( J_\varphi \) is the resolvent operator. Equations of type (4.3) are called the resolvent equations, which are mainly due to Noor [10]. It has been shown in [10] that problems (4.1) and (4.3) are equivalent, provided

\[
u = J_\varphi z
\]

(4.4)

\[
z = u - \rho Au.
\]

(4.5)

Using (4.4) and (4.5), the resolvent equations (4.3) can be written as

\[
u - \rho Au - J_\varphi[u - \rho Au] + \rho AJ_\varphi[u - \rho Au] = 0.
\]

(4.6)

It has been shown in Noor [10] that \( u \in \mathbb{R}^n \) is a solution of (4.1) if and only if \( u \in \mathbb{R}^n \) is a zero of (4.6). This alternative equivalent formulation can be used to suggest and analyze a new dynamical system

\[
\frac{du}{dt} = \lambda \{J_\varphi[u - \rho Au] - \rho AJ_\varphi[u - \rho Au] + \rho Au - u\}, \quad u(t_0) = u_0 \in \mathbb{R}^n,
\]

associated with the variational inclusion (4.1), where \( \lambda \) is a constant. Note that all the results remain true and all proofs work. It suffices to replace \( P_K \) by \( J_\varphi \), since the resolvent operator \( J_\varphi \) is also nonexpansive. It is worth mentioning that problems (4.1) and problem (4.3) include problem (2.1) and (2.5) as well as saddle–point problems, finding a zero of two maximal monotone operators and via the partial inverse notion Weber–Fermat problems, location problems, etc., as special cases, see [8, 11, 12] and the references therein.

5. Conclusion

We have suggested a projected type Wiener–Hopf dynamical system associated with the variational inequality problem. We have proved the global asymptotical stability of the system under the pseudomonotonicity, which is a weaker condition than monotonicity. The suggested dynamical systems can be used in designing
recurrent neural networks for solving variational inequalities and optimization problems. We hope that the theoretical results obtained in this paper can provide a different approach for stability analysis, computation and the analysis and design of new neural networks.

Acknowledgement. The author would like to thank the referee for his/her valuable comments and suggestions.

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