Abstract. In 2000, we showed the Mohebi–Radjabalipour Conjecture under an additional condition, and obtained an invariant subspace theorem concerning subdecomposable operators. In this paper, we obtain a stronger result that the invariant subspace lattice for a class of these operators is rich. The result accurately characterize the invariant subspace lattice for the class of operators.

In [10], Mohebi and Radjabalipour raised the following conjecture.

The Mohebi–Radjabalipour Conjecture (see [10], p. 236). Assume the op­erators $T \in B(X)$ and $B \in B(Z)$ on Banach spaces $X$ and $Z$, and the nonempty open set $G$ in the complex plane $C$, satisfy the following conditions:

1. $qT = Bq$ for some injective $q \in B(X, Z)$ with a closed range $qX$;

2. There exist sequences $\{G(n)\}$ of open sets and $\{M(n)\}$ of invariant sub­spaces of $B$ such that $\overline{G(n)} \subset G(n + 1)$, $G = \bigcup_n G(n)$, $\sigma(B|M(n)) \subset C \setminus G(n)$ and $\sigma(B/M(n)) \subset \overline{G(n)}$, $n = 1, 2, \cdots$;

3. $\sigma(T)$ is dominating in $G$.

Then $T$ has a nontrivial invariant subspace.

It is easy to see that the Mohebi–Radjabalipour Conjecture, if true, will contain the main results of [1–4, 6–8, 10] (and others) as special cases.

In [8], using the S. Brown Technique, we proved the Mohebi–Radjabalipour Conjecture under an additional condition, and obtained the invariant subspace theorem concerning subdecomposable operators as follows:

Theorem A. Assume the operators $T \in B(X)$ and $B \in B(Z)$ on Banach spaces $X$ and $Z$, and the nonempty open set $G$ in $C$, satisfy conditions (1), (2) and (3) in the Mohebi–Radjabalipour Conjecture and the following additional condition:

4. $\{qX + M(n)\}$ is a sequence of closed sets in $Z$.

Then $T$ has a nontrivial invariant subspace.

In this paper, we obtain a stronger result, which accurately characterize the invariant subspace lattice for a class of operators in Theorem A. This result is as follows.

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Theorem 1. Assume the operators $T \in B(X)$ and $B \in B(Z)$ on Banach spaces $X$ and $Z$, and the nonempty open set $G$ in $C$, satisfy the conditions (1), (2), (3) and (4) in Theorem A, then $T$ has a non-trivial invariant subspace. In particular, if the essential spectrum $\sigma_e(T)$ of $T$ is dominating in $G$, then the invariant subspace lattice $\text{Lat}(T)$ for $T$ is rich.

To prove Theorem 1 we first recall some basic notation and facts, and give some lemmas.

We denote by $H^\infty(G)$ the Banach algebra of all bounded analytic functions on $G$ equipped with the norm $\|f\| = \sup\{|f(\lambda)|; \lambda \in G\}$. It is well known that $H^\infty(G)$ is a $w^*$-closed subspace of $L^\infty(G)$ relative to the duality $(L^1(G), L^\infty(G))$ and that a sequence $\{f_k\}$ in $H^\infty(G)$ converges to zero relative to $w^*$-topology if and only if it is norm-bounded and converges to zero uniformly on each compact subset of $G$. In particular, we can identify $H^\infty(G)$ with the dual space of the Banach space $Q = L^1(G)/(H^\infty(G))L$. Since $Q$ is separable, it follows from the characterization of $w^*$-convergent sequences in $H^\infty(G)$ that for each $\lambda \in G$, the point evaluation $E_\lambda : H^\infty(G) \to C, f \to f(\lambda)$, is a $w^*$-continuous linear functional.

A subset $\sigma$ of the complex plane $C$ will be called dominating in the open set $G$ if $\|f\| = \sup\{|f(\lambda)|; \lambda \in \sigma \cap G\}$ holds for all $f \in H^\infty(G)$.

Let $E$ be a Banach space, then $\text{Lat}(E)$ denote the lattice of all closed linear subspace of $E$. Let $X$ be a Banach space and let $T \in B(X)$, then the invariant subspace lattice $\text{Lat}(T)$ for $T$ is called to be rich if there exists an infinite dimensional Banach space $E$ such that $\text{Lat}(T)$ contains a sublattice order isomorphic to $\text{Lat}(E)$.

It is easy to see that if $\sigma(T) \neq \sigma_e(T)$ and $\dim(X) \geq 2$, then $T$ has an invariant subspace.

Throughout the rest of this paper, we shall assume that $X, Z, T, B, q, G, G(n)$ and $M(n)$ are as in Theorem 1, and that $\sigma_e(T)$ is dominating in $G$.

Lemma 2 ([8], Lemma 3). Define $\tilde{B} : qX \to qX$ by $\tilde{B}z = Bz$, and define $\tilde{q} : X \to qX$ by $\tilde{q}x = qx$. Then $\tilde{B} \in B(qX)$, $\tilde{q} \in B(X, qX)$ and

1. For any polynomial $p$, for any vectors $\tilde{z} \in qX$ and $z^* \in Z^*$, we have
   $\langle \tilde{z}, p(\tilde{B}^*)\tilde{z}^* \rangle = \langle \tilde{z}, p(B^*)z^* \rangle$,
   where $\tilde{z}^*$ denotes the restriction of $z^*$ onto $qX$, that is, $\tilde{z}^* = z^*|qX$.

2. For any $x \in X$ and $z^* \in Z^*$, we have $\langle x, q^*z^* \rangle = \langle x, q^*z^* \rangle$ and $\|q^*z^*\| = \|q^*z^*\|$, where $\tilde{z}^* = z^*|qX$.

By [8, p.20], we have $M(n)^{-1}|qX \subset M(n + 1)^{-1}|qX$, $\sigma \left(\tilde{B}^* | (M(n)^{-1}|qX) \right) \subset G(n) \subset G$ for $n = 1, 2, \cdots$. Set

$M(G) = \cup_n (M(n)^{-1}|qX)$.

Then for any $x \in X$, $\tilde{z}^* \in M(G)$, there exists a natural number $n \geq n_0$ such that $\tilde{z}^* \in M(n)^{-1}|qX$, where $n_0$ is defined by [8, Note 2]. Therefore we can define a functional $x \otimes \tilde{z}^* : H^\infty(G) \to C$ by

$x \otimes \tilde{z}^*(f) = \langle x, q^*f(\tilde{B}_{n}^*)\tilde{z}^* \rangle$,
where \( \tilde{B}_n^* \) denotes \( \tilde{B}^*(M(n)^+|qX) \), and \( f(\tilde{B}_n^*) \) is defined by the Riesz–Dunford functional calculus with analytic functions. By [8], \( x \otimes \tilde{z}^* \) is a well-defined \( w^* \)-continuous linear functional which is independent of the particular choice of \( n \).

**Lemma 3** ([8], Lemma 8). Let \( r, s \) be natural numbers. Consider non-negative real numbers \( c_1, c_2, \cdots, c_r \) with \( c_1 + c_2 + \cdots + c_r = 1 \) and complex numbers \( \lambda_1, \lambda_2, \cdots, \lambda_r \in \sigma_{le}(T^*) \cap G \), where \( \sigma_{le}(T^*) \) denotes the left essential spectrum of \( T^* \). If \( a_1, a_2, \cdots, a_s \in X, \tilde{b}^*_1, \tilde{b}^*_2, \cdots, \tilde{b}^*_s \in M(G) \) and \( \varepsilon > 0 \) are arbitrary, then there are vectors \( x \in X, \tilde{z}^* \in M(G) \) such that \( ||x|| \leq 3, ||\tilde{q}^*\tilde{z}^*|| \leq 2 \) and

\[
\begin{align*}
(1) \quad &|| (c_1 E_{\lambda_1} + c_2 E_{\lambda_2} + \cdots + c_r E_{\lambda_r}) - x \otimes \tilde{z}^* || \leq \varepsilon, \\
(2) \quad &|| x \otimes \tilde{b}^*_j || \leq \varepsilon, \quad ||a_j \otimes \tilde{z}^*|| \leq \varepsilon, \quad j = 1, 2, \cdots, s.
\end{align*}
\]

**Lemma 4** (cf. [9], Lemma 1.5). Let \( r, s \) be natural numbers. Consider complex numbers \( c_1, c_2, \cdots, c_r \) with \( ||c_1|| + ||c_2|| + \cdots + ||c_r|| \leq 1 \) and complex numbers \( \lambda_1, \lambda_2, \cdots, \lambda_r \in \sigma_{le}(T^*) \cap G \). If \( a_1, a_2, \cdots, a_s \in X, \tilde{b}^*_1, \tilde{b}^*_2, \cdots, \tilde{b}^*_s \in M(G) \) and \( \varepsilon > 0 \) are arbitrary, then there are vectors \( x \in X, \tilde{z}^* \in M(G) \) such that \( ||x|| \leq 12, ||\tilde{q}^*\tilde{z}^*|| \leq 8 \) and

\[
\begin{align*}
(1) \quad &|| (c_1 E_{\lambda_1} + c_2 E_{\lambda_2} + \cdots + c_r E_{\lambda_r}) - x \otimes \tilde{z}^* || < \varepsilon, \\
(2) \quad &|| x \otimes \tilde{b}^*_j || < \varepsilon, \quad ||a_j \otimes \tilde{z}^*|| < \varepsilon, \quad j = 1, 2, \cdots, s.
\end{align*}
\]

We now denote by \( Q_0 \) the set of those elements \( L \) in \( Q = L^1(G)/H^\infty(G) \) which can be almost factorized in the following sense:

Given a natural number \( s \), vectors \( a_1, a_2, \cdots, a_s \in X, \tilde{b}^*_1, \tilde{b}^*_2, \cdots, \tilde{b}^*_s \in M(G) \) and real number \( \varepsilon > 0 \), there are vectors \( x \in X, \tilde{z}^* \in M(G) \) such that \( \max(||x||, ||\tilde{q}^*\tilde{z}^*||) \leq 1, ||L - x \otimes \tilde{z}^*|| < \varepsilon, \) and

\[
\max\{||x \otimes \tilde{b}^*_j||; j = 1, 2, \cdots, s\} < \varepsilon, \quad \max\{||a_j \otimes \tilde{z}^*||; j = 1, 2, \cdots, s\} < \varepsilon.
\]

It is easy to see that the space \( Q_0 \) is a norm-closed subset of \( Q \).

**Lemma 5.** If the right essential spectrum \( \sigma_{re}(T) \) of \( T \) is dominating in \( G \), then \( \{L \in Q; ||L|| \leq 1/96\} \subset Q_0 \).

**Proof.** Since \( \sigma_{le}(T^*) = \sigma_{re}(T) \) is dominating in \( G \), it follows from [5, Proposition 2.8] that the closed absolutely-convex hull of the set \( \{E_\lambda; \lambda \in \sigma_{le}(T^*) \cap G\} \) is precisely the closed unit ball in \( Q \). On the other hand, by Lemma 4 we have \( \text{aco}(1/96)E_\lambda; \lambda \in \sigma_{le}(T^*) \cap G \subset Q_0 \), where \( \text{aco}(M) \) denotes absolutely-convex hull of \( M \subset C \). Consequently, we obtain

\[
\{L \in Q; ||L|| \leq 1/96\} = \text{aco}\{1/96)E_\lambda; \lambda \in \sigma_{le}(T^*) \cap G\} \subset Q_0.
\]

So the proof of Lemma 5 is complete. \( \square \)

Let \( E \) be a nonempty set and \( m \) a natural number, then we define

\[
E^m = \{(x_1, x_2, \cdots, x_m); x_1, x_2, \cdots, x_m \in E\},
\]

\[
M(m, E) = \{(x_{jk}); x_{jk} \in E, j, k = 1, 2, \cdots, m\}.
\]

We write \( M(\infty, E) \) for the set of all infinite matrices \( (x_{jk})_{j,k \geq 1} \) with coefficients \( x_{jk}(j, k = 1, 2, \cdots) \) in \( E \).
Lemma 6. If the right essential spectrum $\sigma_{re}(T)$ of $T$ is dominating in $G$, then for each matrix $L = (L_{jk})_{j,k \geq 1} \in M(\infty, Q)$, there are sequences $\{x_m\}$ and $\{z_m^*\}$ such that

1. $x_m \in X^m$, $z_m^* \in [M(G)]^m$;

2. for each natural number $j$, the limits
   \[
   x(j) = \lim_{m \to \infty} x_m(j) \in X, \quad x^*(j) = \lim_{m \to \infty} q^* z_m^*(j) \in X^*
   \]
   exist, where $x_m(j)$ and $z_m^*(j)$ denote the $j$th components of $x_m$ and $z_m^*$ respectively;

3. for all natural number $j,k$, we have
   \[
   L_{jk} = \lim_{m \to \infty} x_m(j) \otimes z_m^*(k),
   \]
   where the limit is taken in $Q$.

Proof. Using Lemma 5, the proof of Lemma 6 is similar to that of Proposition 2.6 in [7] and is therefore omitted. $\square$

Proof of Theorem 1. We consider the two cases separately:

Case 1. If $\sigma_{re}(T)$ is not dominating in $G$, then it follows from the condition 3 in Theorem 1 that there is a point $\mu \in (\sigma_e(T) \setminus \sigma_{re}(T)) \cap G$. Therefore $\text{Ker}(\mu - T)$ is an infinite dimensional Banach space. Obviously, the lattice $\text{Lat}(\text{Ker}(\mu - T))$ of all closed linear subspaces of the Banach space $\text{Ker}(\mu - T)$ is a sublattice of $\text{Lat}(T)$. Hence $\text{Lat}(T)$ is rich.

Case 2. If $\sigma_{re}(T)$ is dominating in $G$, then for any given point $\lambda \in G$, by Lemma 6 we can choose sequences $\{x_m\}, \{z_m^*\}, \{x(j)\}$ and $\{x^*(k)\}$ which satisfy conditions (1), (2) and (3) in Lemma 6 with respect to the matrix $L = (L_{jk})_{j,k \geq 1} = (\delta_{jk} E_{\mu})_{j,k \geq 1} \in M(\infty, Q)$, where $\delta_{jk}$ denotes the Kronecker delta. Therefore for any natural numbers $m, k$ there exists a natural number $n = n(m, k)$ such that $z_m^*(k) = z_m^*(k)|qX$ with $z_m^*(k) \in M(n(m, k))$. Consequently for each polynomial $p \in C[z]$ in one complex variable, by Lemma 2 and the condition (1) in Theorem 1 we obtain

\[
\delta_{jk} p(\lambda) = \delta_{jk} E_\lambda(p) = \lim_{m \to \infty} x_m(j) \otimes z_m^*(k)(p)
\]

\[
= \lim_{m \to \infty} \langle x_m(j), q^* p(\bar{B}_{n(m,k)}) z_m^*(k) \rangle
\]

\[
= \lim_{m \to \infty} \langle x_m(j), q^* p(B^*) z_m^*(k) \rangle
\]

\[
= \lim_{m \to \infty} \langle x_m(j), p(T^*) q^* z_m^*(k) \rangle
\]

\[
= \lim_{m \to \infty} \langle p(T) x_m(j), q^* z_m^*(k) \rangle
\]

\[
= \langle p(T) x(j), x^*(k) \rangle
\]

(1)
for all natural numbers \(j, k\). Set

\[
U = \text{span}\{p(T)x(j); p \in \mathbb{C}[z], j = 1, 2, \cdots\},
\]

\[
V = \text{span}\{p(T)(\lambda - T)x(j); p \in \mathbb{C}[z], j = 1, 2, \cdots\}.
\]

Then \(U, V \in \text{Lat}(T)\), and \(V \subset U\), \((\lambda - T)U \subset V\). For every natural number \(k\), define the functional \(\phi_k : U/V \rightarrow \mathbb{C}\) by

\[
\phi_k(x + V) = \langle x, x^*(k) \rangle.
\]

It is easy to see that \(\phi_k\) is a well-defined bounded linear functional on \(U/V\). Also, by (1) we obtain

\[
\phi_k(x(j) + V) = \langle x(j), x^*(k) \rangle = \delta_{jk}.
\]

Therefore \(U/V\) is an infinite dimensional Banach space. If \(\pi : U \rightarrow U/V\) is the canonical quotient map, then the map \(\tau : \text{Lat}(U/V) \rightarrow \text{Lat}(T), S \rightarrow \pi^{-1}(S)\), is a lattice embedding, where \(\text{Lat}(U/V)\) denotes the lattice of all closed linear subspaces of the Banach space \((U/V)\). Consequently \(\text{Lat}(T)\) is rich, and this concludes the proof of Theorem 1.

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**References**


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