PROJECTIVE, FLAT AND MULTIPLICATION MODULES

MAJID M. ALI AND DAVID J. SMITH

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Abstract. In this note all rings are commutative rings with identity and all modules are unital. We consider the behaviour of projective, flat and multiplication modules under sums and tensor products. In particular, we prove that the tensor product of two multiplication modules is a multiplication module and under a certain condition the tensor product of a multiplication module with a projective (resp. flat) module is a projective (resp. flat) module. We investigate a theorem of P.F. Smith concerning the sum of multiplication modules and give a sufficient condition on a sum of a collection of modules to ensure that all these submodules are multiplication. We then apply our result to give an alternative proof of a result of El-Bast and Smith on external direct sums of multiplication modules.

Introduction

Let $R$ be a commutative ring with identity and $M$ a unital $R$–module. Then $M$ is called a multiplication module if every submodule $N$ of $M$ has the form $IM$ for some ideal $I$ of $R$ [5]. Recall that an ideal $A$ of $R$ is called a multiplication ideal if for each ideal $B$ of $R$ with $B \subseteq A$ there exists an ideal $C$ of $R$ such that $B = AC$. Thus multiplication ideals are multiplication modules. In particular, invertible ideals of $R$ are multiplication modules. On the other hand cyclic modules are multiplication, and multiplication modules over local rings are cyclic, [4, Theorem 1] and [2, Theorem 2.1].

Let $R$ be a ring and $K$ and $L$ submodules of an $R$–module $M$. The residual of $K$ by $L$ is $[K : L]$, the set of all $x$ in $R$ such that $xL \subseteq K$. The annihilator of $M$, denoted by ann$(M)$, is $[0 : M]$, and for any $m \in M$ the annihilator of $m$, denoted by ann$(m)$, is $[0 : Rm]$. For an $R$–module $M$, we put $\theta(M) = \sum_{m \in M}[Rm : M]$.

Let $M$ be a multiplication module and $N$ a submodule of $M$. There exists an ideal $I$ of $R$ such that $N = IM$. Note that $I \subseteq [N : M]$ and hence $N = IM \subseteq [N : M]M \subseteq N$, so that $N = [N : M]M$. Note also that $K$ is a multiplication submodule of $M$ if and only if $N \cap K = [N : K]K$ for all submodules $N$ of $M$.

In Section 2, we investigate the tensor product of modules. We show that the tensor product of two multiplication $R$–modules is a multiplication $R$–module, Theorem 2.1. We also prove that if $M$ is a multiplication $R$–module and $P$ is a projective (resp. flat) $R$–module such that ann$(M) \subseteq$ ann$(P)$, then $M \otimes P$ is a projective (resp. flat) $R$–module, Theorems 2.3 and 2.5. We apply Theorem 2.3 to generalize [18, Theorem 11] and show that multiplication modules whose annihilators are generated by an idempotent are in fact projective modules, Corollary 2.4. As 2000 AMS Mathematics Subject Classification: 13C13, 13A15, 15A69.

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a consequence of Theorem 2.5, we give a generalization of [16, Theorem 2.2] and prove that if \( M \) is a multiplication \( R \)-module such that \( \text{ann}(M) \) is a pure ideal, then \( M \) is a flat \( R \)-module, Corollary 2.7.

Section 3 is concerned with the sum of modules. If \( N_\lambda(\lambda \in \Lambda) \) is a collection of submodules of an \( R \)-module \( M \) such that the sum of any two distinct members is multiplication, then \( \sum_{\lambda \in \Lambda} N_\lambda \) need not be multiplication [18]. Theorem 3.1 gives several equivalent conditions under which this sum is multiplication, generalizing P.F. Smith's theorems [18, Theorems 2 and 8(i)].

We also consider the following question: Let \( R \) be a ring and \( N_\lambda(\lambda \in \Lambda) \) a collection of submodules of an \( R \)-module \( M \) such that \( \sum_{\lambda \in \Lambda} N_\lambda \) is multiplication. Is at least one \( N_\lambda \) a multiplication module? We show that the answer is no, and we give a sufficient condition on such a collection to ensure that all of these submodules are multiplication, Theorem 3.2. We apply Theorem 3.2 to give an alternative proof to a result of El–Bast and Smith [8, Theorem 2.2].

All rings are commutative with identity and all modules are unital. For the basic concepts used, we refer the reader to [6], [9], [10], [12] and [13].

1. Preliminaries

P.F. Smith gave necessary and sufficient conditions for the sum of a collection of multiplication modules to be a multiplication module [18, Theorem 2]. We investigate this theorem and prove a strengthened version using alternative methods.

**Lemma 1.1.** Let \( R \) be a ring and \( N_\lambda(\lambda \in \Lambda) \) a collection of submodules of an \( R \)-module \( M \), let \( N = \sum_{\lambda \in \Lambda} N_\lambda \), and let \( A = \sum_{\lambda \in \Lambda}[N_\lambda : N] \). Consider the following statements:

(i) \( N \) is a multiplication module.

(ii) \( H = AH \) for all submodules \( H \) of \( N \).

(iii) \( R = A + \text{ann}(n) \) for all \( n \in N \).

(iv) \( [N : K] + \text{ann}(n) = \sum_{\lambda \in \Lambda}[N_\lambda : K] + \text{ann}(n) \) for all submodules \( K \) of \( M \) and all \( n \in N \).

(v) \( K \cap N = \sum_{\lambda \in \Lambda} K \cap N_\lambda \) for all submodules \( K \) of \( M \).

Then (i) \( \Rightarrow \) (ii) \( \Leftrightarrow \) (iii) \( \Leftrightarrow \) (iv) \( \Rightarrow \) (v). If all \( N_\lambda \) are multiplication modules, then the first four conditions are equivalent.

**Proof.** (i) \( \Rightarrow \) (ii). As \( N \) is multiplication, \( N_\lambda = [N_\lambda : N]N \) for all \( \lambda \in \Lambda \), and hence \( N = AN \). Let \( H \) be any submodule of \( N \). Then \( H = IN \) for some ideal \( I \) of \( R \), and hence

\[
H = IN = I(AN) = A(IN) = AH.
\]

(ii) \( \Rightarrow \) (iii). Let \( n \in N \). Then \( Rn = An \), and hence \( R = A + \text{ann}(n) \).

(iii) \( \Rightarrow \) (ii) is obvious.

(iii) \( \Rightarrow \) (iv). Let \( K \) be any submodule of \( M \). Let \( n \in N \). Clearly

\[
\sum_{\lambda \in \Lambda}[N_\lambda : K] + \text{ann}(n) \subseteq [N : K] + \text{ann}(n).
\]
Conversely, there exist a finite subset $\Lambda'$ of $\Lambda$ and elements $x_\lambda \in [N_\lambda : N](\lambda \in \Lambda')$ and $z \in \text{ann}(n)$ such that $1 = \sum_{\lambda \in \Lambda'} x_\lambda + z$. Let $w \in [N : K] + \text{ann}(n)$. Then $w = w_1 + w_2$ where $w_1 K \subseteq N$ and $w_2 \in \text{ann}(n)$, and hence

$$w = \sum_{\lambda \in \Lambda'} w_1 x_\lambda + (w_1 z + w_2) \in \sum_{\lambda \in \Lambda} [N_\lambda : N] + \text{ann}(n)$$

so that (iv) is satisfied. (iv) $\Rightarrow$ (iii) is clear by taking $K = N$. To complete the proof of the first assertion of the Lemma, it is enough to show that (iii) $\Rightarrow$ (v).

We do this locally. Thus we may assume that $R$ is a local ring. If $N = 0$, the result is trivial. Thus we may suppose that $N \neq 0$. There exists $0 \neq n \in N$ so that $\text{ann}(n) \neq R$, and hence $A = R$. There exists $\mu \in \Lambda$ such that $[N_\mu : N] = R$ and hence $N = N_\mu$. This implies that

$$K \cap N = K \cap N_\mu \subseteq \sum_{\lambda \in \Lambda} K \cap N_\lambda \subseteq K \cap N,$$

so that $K \cap N = \sum_{\lambda \in \Lambda} N_\lambda K \cap N_\lambda$, and the proof of the first part of the lemma is complete. For the last assertion, it is enough to show that (ii) $\Rightarrow$ (i). We first prove that $N_\lambda = [N_\lambda : N] N$ for all $\lambda \in \Lambda$. We do this locally. Suppose that $P$ is a maximal ideal of $R$. We discuss two cases.

Case 1: $A \subseteq P$. Then for all $n \in N$, $Rn = An \subseteq Pn$, and hence $(Rn)_P = P_P(Rn)_P$, and by Nakayama's Lemma we have that $(Rn)_P = 0_P$. It follows that $N_P = 0_P$, and hence $(N_\lambda)_P = 0_P$ for all $\lambda \in \Lambda$. Both sides of the equality $N_\lambda = [N_\lambda : N] N$ collapse to $0_P$.

Case 2: $A \not\subseteq P$. There exists $\mu \in \Lambda$ and $p \in P$ such that $(1-p)N \subseteq N_\mu$. It follows that $N_P = (N_\mu)_P$, and hence for all $\lambda \in \Lambda$,

$$[N_\lambda : N_\mu]_P \subseteq [N_\lambda : (1-p)N]_P = [[N_\lambda : N] : (R(1-p))]_P = [[N_\lambda : N] : (R(1-p))P] = [N_\lambda : N]_P \subseteq [N_\lambda : N_\mu]_P,$$

so that $[N_\lambda : N_\mu]_P = [N_\lambda : N]_P$. This implies that

$$[N_\lambda : N]_PPN_P = [N_\lambda : N_\mu]_PP(N_\mu)P = (N_\lambda \cap N_\mu)_P = (N_\lambda)_P \cap (N_\mu)_P = (N_\lambda)_P.$$

From the two cases, we infer that the equation $N_\lambda = [N_\lambda : N] N$ is true locally and hence globally. Now assume that $K$ is a submodule of $M$. Then by (v) of the lemma,

$$K \cap N = \sum_{\lambda \in \Lambda} K \cap N_\lambda = \sum_{\lambda \in \Lambda} [K : N_\lambda]N_\lambda$$

and

$$= \sum_{\lambda \in \Lambda} [K : N_\lambda] [N_\lambda : N]N \subseteq [K : N]N \subseteq K \cap N,$$

so that $K \cap N = [K : N]N$, and hence $N$ is a multiplication module. □
Note that the implication \((v) \Rightarrow (iii)\) in the above Lemma is not true. Let \(R\) be a Prüfer domain but not Noetherian. For example, take any non–Noetherian valuation domain. There is a maximal ideal \(P\) of \(R\) which is not finitely generated. \(P\) is not multiplication, and hence \(\theta(P) \neq R\), \([2, \text{Theorem 2.1}]\) and \([3, \text{Theorem 1}]\). However, for every non–zero ideal \(I\) of \(R\), \(I \cap P = \sum_{p \in P} I \cap Rp\). The same example shows that \((iii)\) does not imply \((i)\), by taking \(N_1 = N_2 = P\).

The following consequence is straightforward.

**Corollary 1.2.** Let \(R\) be a ring and \(M\) an \(R\)-module. Then \(M\) is multiplication if and only if \(\theta(M) + \text{ann}(m) = R\) for all \(m \in M\). Consequently, \(M\) is multiplication if and only if \(Rm = \theta(M)m\) for all \(m \in M\).

Let \(M\) be an \(R\)-module and let \(S = \{m_\lambda | \lambda \in \Lambda\}\) be a set of generators for \(M\). Let \(F\) be the free \(R\)-module generated by the set \(S' = \{x_\lambda | \lambda \in \Lambda\}\) and let \(\pi : F \rightarrow M\) be defined on \(S'\) by \(\pi(x_\lambda) = m_\lambda\) and then extended to \(F\) linearly. Let \(K\) be the kernel of \(\pi\). Then the following exact sequence is called a presentation for \(M\):

\[
0 \rightarrow K \rightarrow F \overset{\pi}{\rightarrow} M \rightarrow 0.
\]

The following characterizations of projective and flat modules can be found in [7] and [15].

**Lemma 1.3.** Let \(M\) an \(R\)-module. Then \(M\) is projective if and only if for each presentation

\[
0 \rightarrow K \rightarrow F \overset{\pi}{\rightarrow} M \rightarrow 0
\]

for \(M\), there exists an \(R\)-homomorphism \(\theta : F \rightarrow F\) such that

(i) \(\pi \circ \theta = \pi\) and

(ii) \(\ker \theta = \ker \pi\).

**Lemma 1.4.** Let \(M\) be an \(R\)-module. Then \(M\) is flat if and only if for each presentation

\[
0 \rightarrow K \rightarrow F \overset{\pi}{\rightarrow} M \rightarrow 0
\]

and each \(u \in K\), there exists an \(R\)-homomorphism \(\theta_u : F \rightarrow F\) such that

(i) \(\pi \circ \theta_u = \pi\) and

(ii) \(\theta_u(u) = 0\).

We also need the next result about flat modules. The proof can be found in [7].

**Lemma 1.5.** Let \(M\) be a flat \(R\)-module with a presentation as above. Let \(u_i \in K, 1 \leq i \leq n\) and let \(\theta_i\) be the corresponding homomorphisms given above. Then there exists a homomorphism \(\theta : F \rightarrow F\) such that

(i) \(\pi \circ \theta = \pi\) and

(ii) \(\theta(u_i) = 0\) for all \(1 \leq i \leq n\).

2. Tensor Products

Let \(R\) be a ring. It is well known that if \(I\) and \(J\) are projective (resp. flat, multiplication) ideals of \(R\), then \(IJ\) is projective (resp. flat, multiplication), [3] and [15]. If \(M\) is a multiplication \(R\)-module and \(I\) is a multiplication ideal, then \(IM\) is a multiplication \(R\)-module, [8]. If \(M\) is a finitely generated multiplication \(R\)-module and \(I\) is a projective (resp. flat) ideal of \(R\) such that \(\text{ann}(M) \cap I = 0\), then \(IM\) is a projective (resp. flat) \(R\)-module [14]. It is also well–known [6] that the tensor product of two projective (resp. flat) \(R\)-modules is a projective (resp. flat) \(R\)-module. We now prove that the tensor product of two multiplication \(R\)-modules is a multiplication \(R\)-module. We also show that if \(M\) is a multiplication \(R\)-module
and $N$ is a projective (resp. flat) $R$-module such that $\text{ann } M \subseteq \text{ann } N$, then the tensor product $M \otimes N$ is a projective (resp. flat) $R$-module.

**Theorem 2.1.** Let $R$ be a ring and $M_1$ and $M_2$ multiplication $R$-modules. Then $M_1 \otimes M_2$ is a multiplication $R$-module.

**Proof.** We first prove that $\theta(M_1)\theta(M_2) \subseteq \theta(M_1 \otimes M_2)$. Let $m_1 \in M_1$ and $m_2 \in M_2$. If $x \in [Rm_1 : M_1]$ and $y \in [Rm_2 : M_2]$, then $xM_1 \subseteq Rm_1$ and $yM_2 \subseteq Rm_2$. It follows that
\[
x y(M_1 \otimes M_2) = xM_1 \otimes yM_2 \subseteq Rm_1 \otimes Rm_2 = R(m_1 \otimes m_2),
\]
and hence $xy \in [R(m_1 \otimes m_2) : M_1 \otimes M_2]$. This implies that $[Rm_1 : M_1][Rm_2 : M_2] \subseteq [R(m_1 \otimes m_2) : M_1 \otimes M_2]$ and hence $\theta(M_1)\theta(M_2) \subseteq \theta(M_1 \otimes M_2)$. Next, since each of $M_1$ and $M_2$ is a multiplication $R$-module, we infer from Corollary 1.2 that $Rm_1 = \theta(M_1)m_1$ and $Rm_2 = \theta(M_2)m_2$. Hence
\[
R(m_1 \otimes m_2) = Rm_1 \otimes Rm_2 = \theta(M_1)m_1 \otimes \theta(M_2)m_2
= \theta(M_1)\theta(M_2)(m_1 \otimes m_2)
\subseteq \theta(M_1 \otimes M_2)(m_1 \otimes m_2)
\]
so that $R(m_1 \otimes m_2) = \theta(M_1 \otimes M_2)(m_1 \otimes m_2)$. From this we obtain that $Rx = \theta(M_1 \otimes M_2)x$ for all $x \in M_1 \otimes M_2$, and the result follows by Corollary 1.2. □

The next lemma should be compared with [16, Lemma 3.6].

**Lemma 2.2.** Let $R$ be a ring and $M$ a multiplication $R$-module. Let $\{m_\lambda | \lambda \in \Lambda \}$ be a set of generators for $M$. Then for each $\lambda \in \Lambda$, there exist a finite subset $\Lambda_\lambda$ of $\Lambda$ and elements $t_{\mu \alpha} \in R$ with $\mu \in \Lambda_\lambda$ (depending on $\lambda$) and $\alpha \in \Lambda$ such that the following conditions are satisfied:

1. $m_\lambda = \left( \sum_{\mu \in \Lambda_\lambda} t_{\mu \alpha} \right) m_\lambda$.
2. $t_{\mu \alpha} m_\beta = t_{\mu \beta} m_\alpha$ for all $\alpha, \beta \in \Lambda$ and all $\mu \in \Lambda_\lambda$.
3. $m_\lambda = \sum_{\mu \in \Lambda_\lambda} t_{\mu \lambda} m_\mu$.

**Proof.** Let $\lambda \in \Lambda$. It follows by Corollary 1.2 that $\theta(M) + \text{ann}(m_\lambda) = R$. There exists a finite subset $\Lambda_\lambda$ of $\Lambda$ such that
\[
\sum_{\mu \in \Lambda_\lambda} [Rm_\mu : M] + \text{ann}(m_\lambda) = R.
\]

It follows by [16, Lemma 3.5] that there exist $c_{\mu \lambda} \in [Rm_\mu : M]$, $d_\mu$, $d_\lambda \in R$ and $w_\lambda \in \text{ann}(m_\lambda)$ such that
\[
\sum_{\mu \in \Lambda_\lambda} d_\mu c_{\mu \lambda}^2 + d_\lambda w_\lambda^2 = 1.
\]
As $c_{\mu \lambda} \in [Rm_\mu : M]$, $c_{\mu \lambda} M \subseteq Rm_\mu$, it follows that for all $\alpha \in \Lambda$ there exists $c_{\mu \alpha} \in R$ such that $c_{\mu \lambda} m_\alpha = c_{\mu \alpha} m_\mu$. Put $t_{\mu \alpha} = d_\mu c_{\mu \alpha}$. Then
\[
\left( \sum_{\mu \in \Lambda_\lambda} d_\mu c_{\mu \alpha}^2 \right) m_\lambda + d_\lambda w_\lambda^2 m_\lambda = m_\lambda.
\]
and hence \( m_\lambda = \left( \sum_{\mu \in \Lambda_\lambda} t_{\mu \lambda} \right) m_\lambda \). This proves (i). Let \( \alpha, \beta \in \Lambda \) and \( \mu \in \Lambda_\lambda \). Then
\[
t_{\mu \alpha} m_\beta = d_\mu c_{\mu \alpha} c_{\mu \beta} m_\beta = d_\mu c_{\mu \beta} c_{\mu \alpha} m_\mu
\]
and
\[
t_{\mu \beta} m_\alpha = d_\mu c_{\mu \beta} c_{\mu \alpha} m_\alpha = d_\mu c_{\mu \alpha} c_{\mu \beta} m_\mu
\]
and hence (ii) is satisfied. By (i) and (ii), it is now clear that
\[
m_\lambda = \sum_{\mu \in \Lambda_\lambda} t_{\mu \lambda} m_\mu = \sum_{\mu \in \Lambda_\lambda} t_{\mu \lambda} m_\mu.
\]

\[\square\]

Theorem 2.3. Let \( R \) be a ring and \( M \) a multiplication \( R \)-module. If \( P \) is a projective \( R \)-module such that \( \text{ann} M \subseteq \text{ann} P \), then \( M \otimes P \) is a projective \( R \)-module.

Proof. Let \( \{m_\lambda|\lambda \in \Lambda\} \) be a set of generators for \( M \). Let \( F_1 \) be the free \( R \)-module generated by \( \{x_\lambda|\lambda \in \Lambda\} \) and define \( \pi_1 : F_1 \to M \) by \( \pi_1(x_\lambda) = m_\lambda \) for all \( \lambda \in \Lambda \). Define \( \theta_1 : F_1 \to F_2 \) by \( \theta_1(x_\lambda) = \sum_{\mu \in \Lambda_\lambda} t_{\mu \lambda} x_\mu \) where \( \Lambda_\lambda \) and \( t_{\mu \lambda} \) are as defined in Lemma 2.2. It follows that for all \( \lambda \in \Lambda \),
\[
(\pi_1 \circ \theta_1)(x_\lambda) = \pi_1(\theta_1(x_\lambda)) = \pi_1 \left( \sum_{\mu \in \Lambda_\lambda} t_{\mu \lambda} x_\mu \right) = \sum_{\mu \in \Lambda_\lambda} t_{\mu \lambda} \pi_1(x_\mu) = \sum_{\mu \in \Lambda_\lambda} t_{\mu \lambda} m_\mu = m_\lambda = \pi_1(x_\lambda).
\]
Hence \( \pi_1 \circ \theta_1 = \pi_1 \).

Assume that \( P \) is generated by \( \{p_k|k \in I\} \). Let \( F_2 \) be the free \( R \)-module generated by \( \{y_k|k \in I\} \). Consider the exact sequence
\[
0 \to K_2 \to F_2 \stackrel{\pi_2}{\to} P \to 0,
\]
where \( \pi_2 : F_2 \to P \) is defined by \( \pi_2(y_k) = p_k \) for all \( k \in I \), and \( K_2 = \ker \pi_2 \). It follows by Lemma 1.3 that there exists an \( R \)-homomorphism \( \theta_2 : F_2 \to F_2 \) such that (i) \( \pi_2 \circ \theta_2 = \pi_2 \) and (ii) \( \ker \pi_2 = \ker \theta_2 \). The \( R \)-module \( M \otimes P \) is generated by the set \( \{m_\lambda \otimes p_k|\lambda \in \Lambda, k \in I\} \). Let \( F = F_1 \otimes F_2 \) and consider the exact sequence
\[
0 \to K \to F \stackrel{\pi}{\to} M \otimes P \to 0,
\]
where \( \pi = \pi_1 \otimes \pi_2 \). Define \( \theta : F \to F \) by \( \theta = \theta_1 \otimes \theta_2 \). Then \( \pi(x_\lambda \otimes y_k) = \pi_1(x_\lambda) \otimes \pi_2(y_k) \) and \( \theta(x_\lambda \otimes y_k) = \theta_1(x_\lambda) \otimes \theta_2(y_k) \) for all \( \lambda \in \Lambda \) and all \( k \in I \). To prove that \( M \otimes P \) is a projective \( R \)-module, we show that \( \theta \) satisfies the conditions of Lemma 1.3. Let \( k \in I \) and let \( \theta_2(y_k) = \sum_{l=1}^n r_{lk} y_l \). Then for all \( \lambda \in \Lambda \),
\[
(\pi \circ \theta)(x_\lambda \otimes y_k) = \pi(\theta_1(x_\lambda) \otimes \theta_2(y_k)) = \pi_1(\theta_1(x_\lambda)) \otimes \pi_2(\theta_2(y_k)) = \pi_1(x_\lambda) \otimes \pi_2(y_k) = \pi(x_\lambda \otimes y_k),
\]
i.e. \( \pi \circ \theta = \pi \). For the second condition of Lemma 1.3, we obtain from [10, Theorem 7.7] that \( \ker \pi \) is generated by elements \( x \otimes y \) where either \( x \in \ker \pi_1 \) or \( y \in \ker \pi_2 \). We discuss two cases.
Case 1: \( y \in \ker \pi_2 \). Then \( y \in \ker \theta_2 \), and hence \( \theta_2(y) = 0 \). It follows that
\[
\theta(x \otimes y) = \theta_1(x) \otimes 0 = 0,
\]
and hence \( x \otimes y \in \ker \theta \).

Case 2: \( x \in \ker \pi_1 \). There exists a finite subset \( \Lambda' \) of \( \Lambda \) such that
\[
x = \sum_{\alpha \in \Lambda'} w_\alpha x_\alpha \text{ with } w_\alpha \in R.
\]
Hence
\[
0 = \pi_1(x) = \pi_1 \left( \sum_{\alpha \in \Lambda'} w_\alpha x_\alpha \right) = \sum_{\alpha \in \Lambda'} w_\alpha \pi_1(x_\alpha) = \sum_{\alpha \in \Lambda'} w_\alpha m_\alpha.
\]
Next,
\[
\theta(x \otimes y) = \theta_1(x) \otimes \theta_2(y) = \theta_1 \left( \sum_{\alpha \in \Lambda'} w_\alpha x_\alpha \right) \otimes \theta_2(y)
\]
\[
= \left( \sum_{\alpha \in \Lambda'} w_\alpha \sum_{\mu \in \Lambda_\alpha} t_{\mu \alpha} x_\mu \right) \otimes \theta_2(y) \quad (\ast)
\]
where \( \Lambda_\alpha \) and \( t_{\mu \alpha} \) are as defined in Lemma 2.2. Since \( \sum_{\alpha \in \Lambda'} w_\alpha m_\alpha = 0 \), we have for all \( \beta \in \Lambda \),
\[
\left( \sum_{\alpha \in \Lambda'} t_{\mu \alpha} w_\alpha \right) m_\beta = \sum_{\alpha \in \Lambda'} w_\alpha t_{\mu \alpha} m_\beta = \sum_{\alpha \in \Lambda'} w_\alpha t_{\mu \beta} m_\alpha = 0.
\]
Hence \( \sum_{\alpha \in \Lambda'} t_{\mu \alpha} w_\alpha \in \ann(m_\beta) \), and therefore \( \sum_{\alpha \in \Lambda'} t_{\mu \alpha} w_\alpha \in \ann M \subseteq \ann P \). It follows that \( \sum_{\alpha \in \Lambda'} t_{\mu \alpha} w_\alpha p = 0 \) for all \( p \in P \), and hence \( \sum_{\alpha \in \Lambda'} t_{\mu \alpha} w_\alpha \pi_2(y) = 0 \). So \( \sum_{\alpha \in \Lambda'} t_{\mu \alpha} w_\alpha y \in \ker \pi_2 = \ker \theta_2 \), and
\[
0 = \theta_2 \left( \sum_{\alpha \in \Lambda'} t_{\mu \alpha} w_\alpha y \right) = \sum_{\alpha \in \Lambda'} t_{\mu \alpha} w_\alpha \theta_2(y).
\]
Hence from \( \ast \) we have that \( \theta(x \otimes y) = 0 \), and this completes the proof of the theorem. \( \square \)

Although not all projective \( R \)-modules are multiplication, it is well known [19] that every projective ideal is multiplication. Moreover, it is proved [15] that if \( I \) is a projective ideal then \( \ann I = \ann L \) for some idempotent ideal \( L \) of \( R \). More is known for the finitely generated case. For example, it is shown [16] that a finitely generated ideal \( I \) of \( R \) is projective if and only if \( I \) is multiplication and \( \ann I = Re \) for some idempotent element \( e \). This result has been extended for modules [18]: If \( M \) is a finitely generated multiplication \( R \)-module such that \( \ann M = Re \) for some idempotent \( e \), then \( M \) is projective. The next consequence of Theorem 2.3 is a generalization of this last result to multiplication modules (not necessarily finitely generated).

**Corollary 2.4.** Let \( R \) be a ring and \( M \) a multiplication \( R \)-module such that \( \ann M = Re \) for some idempotent \( e \). Then \( M \) is projective.
Proof. As \( \text{ann } M = Re = \text{ann}(1 - e) \) and \( R(1 - e) \) is a projective ideal of \( R \), we infer from Theorem 2.3 that \( M \otimes R(1 - e) \) is a projective \( R \)-module. But \( R = Re + R(1 - e) \). Thus

\[
M \otimes R = M \otimes Re + M \otimes R(1 - e).
\]

Finally \( M \otimes Re = eM \otimes R = 0 \otimes R = 0 \), and \( M \otimes R \cong M \). It follows that \( M \cong M \otimes R(1 - e) \), and hence \( M \) is projective. \( \square \)

Theorem 2.5. Let \( R \) be a ring and \( M \) a multiplication \( R \)-module. If \( P \) is a flat \( R \)-module such that \( \text{ann } M \subseteq \text{ann } P \), then \( M \otimes P \) is a flat \( R \)-module.

Proof. Assume that \( F_1, F_2 \) and \( F \) are as in the proof of Theorem 2.3. Also assume that \( \pi_1, \pi_2, \pi \) and \( \theta_1 \) are as defined before. Let \( x \otimes y \) be a generator in \( \ker \pi \). To show that \( M \otimes P \) is flat, we need to show that there exists an \( R \)-homomorphism \( \theta_{x \otimes y} : F \to F \) such that (i) \( \pi \circ \theta_{x \otimes y} = \pi \) and (ii) \( \theta_{x \otimes y}(x \otimes y) = 0 \). We discuss two cases.

Case 1: \( y \in \ker \pi_2 \). As \( P \) is flat, it follows by Lemma 1.4 that there exists an \( R \)-homomorphism \( \theta_y : F_2 \to F_2 \) such that (i) \( \pi_2 \circ \theta_y = \pi_2 \) and (ii) \( \theta_y(y) = 0 \). Define \( \theta_{x \otimes y} : F \to F \) by \( \theta_{x \otimes y} = \theta_1 \otimes \theta_y \). Then

\[
\pi \circ \theta_{x \otimes y} = (\pi_1 \otimes \pi_2) \circ (\theta_1 \otimes \theta_y) = (\pi_1 \circ \theta_1) \otimes (\pi_2 \circ \theta_y) = \pi_1 \otimes \pi_2 = \pi.
\]

As \( \theta_y(y) = 0 \), \( \theta_{x \otimes y}(x \otimes y) = 0 \).

Case 2: \( x \in \ker \pi_1 \). There exists a finite subset \( \Lambda' \) of \( \Lambda \) such that

\[
x = \sum_{\alpha \in \Lambda'} w_\alpha x_\alpha,
\]

where \( w_\alpha \in R \). It follows that

\[
0 = \pi_1(x) = \pi_1 \left( \sum_{\alpha \in \Lambda'} w_\alpha x_\alpha \right) = \sum_{\alpha \in \Lambda'} w_\alpha \pi_1(x_\alpha) = \sum_{\alpha \in \Lambda'} \sum_{\mu \in \Lambda_\alpha} w_\alpha t_{\mu \alpha} x_\mu,
\]

and

\[
\theta_1(x) = \theta_1 \left( \sum_{\alpha \in \Lambda'} w_\alpha x_\alpha \right) = \sum_{\alpha \in \Lambda'} w_\alpha \theta_1(x_\alpha) = \sum_{\alpha \in \Lambda'} \sum_{\mu \in \Lambda_\alpha} w_\alpha t_{\mu \alpha} x_\mu,
\]

where \( \Lambda_\alpha \) and \( t_{\mu \alpha} \) are as defined in Lemma 2.2. As we have seen in the proof of Theorem 2.3, \( \sum_{\alpha \in \Lambda'} w_\alpha t_{\mu \alpha} \in \text{ann } M \subseteq \text{ann } P \), and hence \( \sum_{\alpha \in \Lambda'} w_\alpha t_{\mu \alpha} p = 0 \) for all \( p \in P \). This implies \( \pi_2 \left( \sum_{\alpha \in \Lambda'} w_\alpha t_{\mu \alpha} y \right) = 0 \), i.e. \( \sum_{\alpha \in \Lambda'} w_\alpha t_{\mu \alpha} y \in \ker \pi_2 \). We infer from Lemma 1.5 that there exists an \( R \)-homomorphism \( \tilde{\theta} : F_2 \to F_2 \) such that (i) \( \pi_2 \circ \tilde{\theta} = \pi_2 \) and (ii) \( \tilde{\theta} \left( \sum_{\alpha \in \Lambda'} w_\alpha t_{\mu \alpha} y \right) = 0 \), and hence \( \sum_{\alpha \in \Lambda'} w_\alpha t_{\mu \alpha} y \in \ker \tilde{\theta} \). Define \( \theta_{x \otimes y} : F \to F \) by \( \theta_{x \otimes y} = \theta_1 \otimes \tilde{\theta} \). As \( \tilde{\theta}(y) \in F_2 \), put \( \hat{\theta}(y) = \sum_{l=1}^n r_l y_l \). Hence

\[
\theta_{x \otimes y}(x \otimes y) = \theta_1(x) \otimes \hat{\theta}(y) = \left( \sum_{\alpha \in \Lambda'} \sum_{\mu \in \Lambda_\alpha} w_\alpha t_{\mu \alpha} x_\mu \right) \otimes \sum_{l=1}^n r_l y_l
\]

\[
= \sum_{\alpha \in \Lambda'} \sum_{\mu \in \Lambda_\alpha} \sum_{l=1}^n w_\alpha t_{\mu \alpha} r_l (x_\mu \otimes y_l) \quad (\ast)
\]

Now,

\[
0 = \hat{\theta} \left( \sum_{\alpha \in \Lambda'} w_\alpha t_{\mu \alpha} y \right) = \sum_{\alpha \in \Lambda'} w_\alpha t_{\mu \alpha} \tilde{\theta}(y) = \sum_{\alpha \in \Lambda'} \sum_{l=1}^n w_\alpha t_{\mu \alpha} r_l y_l \quad (\ast\ast)
\]
But $F_2$ is a free $R$-module. Thus for all $1 \leq l \leq n$, the coefficient of $y_l$ in (***) is zero. Hence

$$\sum_{\alpha \in \Lambda} \sum_{l=1}^{n} w_{\alpha} t_{\mu} r_l = 0.$$ 

It follows from (*) that $\theta_{x \otimes y}(x \otimes y) = 0$. Also,

$$\pi \circ \theta_{x \otimes y} = (\pi_1 \otimes \pi_2) \circ (\theta_1 \otimes \hat{\theta}) = (\pi_1 \circ \theta_1) \otimes (\pi_2 \circ \hat{\theta}) = \pi_1 \otimes \pi_2 = \pi,$$

and this finishes the proof of the theorem. □

Following [17, p. 100], a submodule $N$ of an $R$-module $M$ is called a pure submodule if $\alpha M = \alpha N$ for all $\alpha \in \mathbb{R}$. In particular, an ideal $I$ of $R$ is pure if $Rx = Ix$ for every $x \in I$. It is known that every pure ideal of a ring $R$ is either locally zero or locally $R$. For, take $M$ to be any maximal ideal of $R$. If $I \subseteq M$, then for all $x \in I$, $(Rx)_M = I_M(Rx)_M \subseteq M_M(Rx)_M$, and hence $(Rx)_M = M_M(Rx)_M$, and by Nakayama’s Lemma, $(Rx)_M = 0_M$, and hence $I_M = 0_M$. On the other hand, if $I \not\subseteq M$, then $I \cap (R \setminus M) \neq \emptyset$, and hence $I_M = R_M$.

It is well known that if $I$ is a finitely generated multiplication ideal with pure annihilator, then $I$ is a flat ideal [16, Theorem 2.2]. We generalize this result to multiplication modules (not necessarily finitely generated). We first give a lemma.

**Lemma 2.6.** Let $R$ be a ring and $M$ a multiplication $R$-module such that $\text{ann} M$ is a finitely generated pure ideal. Then $M$ is a flat module.

**Proof.** Let $\text{ann} M = \sum_{i=1}^{n} Ra_i$. Then for all $i$, there exists $y_i \in \text{ann} M$ such that $a_i(1 - y_i) = 0$. Put

$$1 - y = (1 - y_1)(1 - y_2) \cdots (1 - y_n).$$

Then $y \in \text{ann} M$, and $a_i(1 - y) = 0$. It follows that $a_i \in \text{ann}(1 - y)$ for all $i$, and hence $\text{ann} M \subseteq \text{ann}(1 - y)$. On the other hand, as $yM = 0$, we infer that $M = (1 - y)M$ and hence $\text{ann}(1 - y) \subseteq \text{ann} M$. This gives that $\text{ann} M = \text{ann}(1 - y)$ and that $y$ is idempotent. Now, $\text{ann}(1 - y)$ is a pure ideal, and by [16, Theorem 2.2] $R(1 - y)$ is a flat ideal of $R$. It follows by Theorem 2.5 that $M \otimes R(1 - y)$ is a flat module. But

$$M \otimes R = M \otimes Ry + M \otimes R(1 - y)$$

and $M \otimes R \cong M$ and $M \otimes Ry = yM \otimes R = 0$, thus $M \cong M \otimes R(1 - y)$, and $M$ is a flat module. □

**Corollary 2.7.** Let $R$ be a ring and $M$ a multiplication $R$-module. Suppose that $\text{ann} M$ is a pure ideal. Then $M$ is a flat module.

**Proof.** Assume $P$ is a maximal ideal of $R$. By [8, Theorem 1.2] either $M_P = 0_P$ in which case it is obvious that $M_P$ is a flat $R_P$-module, or there exist $p \in P$ and $m \in M$ such that $(1 - p)M \subseteq Rm$. It follows that $(1 - p) M \text{ann}(m) = 0$, and hence $(1 - p) \text{ann}(m) \subseteq \text{ann} M$. This implies that

$$\text{ann}(m)_P \subseteq (\text{ann} M)_P \subseteq \text{ann}(M_P) \subseteq \text{ann}(Rm)_P = \text{ann}(m)_P,$$

so that $\text{ann}(M_P) = \text{ann}(M)_P$, see also [3, Theorem 2.1]. As $\text{ann} M$ is a pure ideal of $R$, we have that $\text{ann}(M_P)$ is a pure ideal of $R_P$. Since every pure ideal is idempotent and hence multiplication, we infer that $\text{ann}(M_P)$ is a principal ideal of
By Lemma 2.6, $M_p$ is again a flat $R_p$-module. Hence, $M$ is locally flat and therefore $M$ is flat \([6]\).

We make three observations. First, as a finitely generated pure ideal is a principal ideal generated by an idempotent, Corollary 2.4 shows that a multiplication $R$-module $M$ whose annihilator is a finitely generated pure ideal is projective. But every projective module is flat, \([6]\), \([7]\), \([13]\). Thus $M$ is a flat module. This gives a different proof of Lemma 2.6. Second, from the proof of Corollary 2.7 it is obvious that a multiplication module whose annihilator is an idempotent ideal is a flat module. Third, $Q$ is a flat $Z$-module, but is not multiplication, which shows that the converse of Corollary 2.7 is not true in general.

Our final result of this chapter may be compared with \([14, \text{Theorems 2.1 and 2.2}]\). The proof is now straightforward.

**Corollary 2.8.** Let $R$ be a ring and $M$ a multiplication $R$-module. If $I$ is a projective (resp. flat) ideal of $R$ such that $\text{ann} M \cap I = 0$, then $M \otimes I$ is a projective (resp. flat) $R$-module.

### 3. Sums of Multiplication Modules

Let $Z$ be the ring of integers and $p$ a prime number. Consider the Prüfer $p$-group $Z_{p^\infty}$ as a $Z$-module. Note that $Z_p$ can be identified with a submodule of $Z_{p^2}$. In fact, $pZ_{p^2} = Z_p$. More generally, for all $m, k \geq 1$ with $m > k$, $Z_{p^k}$ can be identified with a submodule of $Z_{p^m}$. Hence

$$Z_p \subset Z_{p^2} \subset \cdots \subset Z_{p^\infty}$$

and $Z_{p^\infty} = \bigcup_{k \geq 1} Z_{p^k}$. These $Z_{p^k}$ are the only nonzero submodules of $Z_{p^\infty}$, and each is a cyclic (hence multiplication) submodule. Also, for all $m, k \geq 1$, $Z_{p^k} + Z_{p^m} = Z_{p^r}$, where $r = \max\{k, m\}$, so that the sum of any two submodules of $Z_{p^\infty}$ is multiplication. We note however that $Z_{p^\infty}$ is neither finitely generated nor multiplication. The following theorem gives several equivalent conditions for the sum of a collection of submodules in which the sum of any two distinct members is multiplication to be a multiplication module. It generalizes \([18, \text{Theorems 2 and 8(i)}]\).

**Theorem 3.1.** Let $R$ be a ring and $M$ an $R$-module. Let $N_\lambda (\lambda \in \Lambda)$ be a collection of submodules of $M$ such that $N_\lambda + N_\mu$ is multiplication for all $\lambda \neq \mu$. Let $N = \sum_{\lambda \in \Lambda} N_\lambda$ and $A = \sum_{\lambda \in \Lambda} [N_\lambda : N]$. Then the following conditions are equivalent:

(i) \[ A + \text{ann}(n) = R \text{ for all } n \in N. \]

(ii) \[ N_\lambda = [N_\lambda : N]N \text{ for all } \lambda \in \Lambda. \]

(iii) For every finite sum $H$ of two or more modules $N_\lambda$, $H$ is multiplication and $H = [H : N]N$.

(iv) $N$ is multiplication.

**Proof.** (i) $\implies$ (ii). Let $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$. By Lemma 1.1, $N_\lambda + N_\mu = A(N_\lambda + N_\mu)$. Let $\lambda \in \Lambda$ and $x \in N_\lambda$. Suppose that

$$K = \{ r \in R | rx \in [N_\lambda : N]N \}.$$
Assume that $K \neq R$. Then there exists a maximal ideal $P$ of $R$ such that $K \subseteq P$. We discuss two cases.

Case 1: $A \subseteq P$. Then for all $\mu \neq \lambda$, $N_{\lambda} + N_{\mu} = A(N_{\lambda} + N_{\mu}) \subseteq P(N_{\lambda} + N_{\mu}) \subseteq N_{\lambda} + N_{\mu}$ so that $N_{\lambda} + N_{\mu} = P(N_{\lambda} + N_{\mu})$. As $N_{\lambda} + N_{\mu}$ is multiplication, we infer that $Rx = I(N_{\lambda} + N_{\mu})$ for some ideal $I$ of $R$, and hence
\[ Rx = I(N_{\lambda} + N_{\mu}) = IP(N_{\lambda} + N_{\mu}) = P(I(N_{\lambda} + N_{\mu})) = Px. \]

There exists $p \in P$ such that $(1 - p)x = 0$. It follows that $(1 - p) \in K \subseteq P$, a contradiction.

Case 2: $A \not\subseteq P$. There exists $\mu \in \Lambda$ such that $[N_{\mu} : N] \not\subseteq P$, and hence there exists $q \in P$ such that $(1 - q)N \subseteq N_{\mu}$. It follows that $(1 - q)N_{\lambda} \subseteq N_{\lambda} + N_{\mu}$ and hence there exists an ideal $J$ of $R$ such that $(1 - q)N_{\lambda} = J(N_{\lambda} + N_{\mu})$. Next
\[ (1 - q)JN = J(1 - q)N \subseteq J(N_{\lambda} + N_{\mu}) = (1 - q)N_{\lambda} \subseteq N_{\lambda} \]
so that $(1 - q)J \subseteq [N_{\lambda} : N]$. This implies that
\[ (1 - q)^2 x \in (1 - q)^2 N_{\lambda} = (1 - q)J(N_{\lambda} + N_{\mu}) \subseteq (1 - q)JN \subseteq [N_{\lambda} : N]. \]
But this implies $(1 - q)^2 \in K \subseteq P$, a contradiction. Thus $K = R$ and $x \in [N_{\lambda} : N]N$, that is $N_{\lambda} \subseteq [N_{\lambda} : N]N$. Since the other inclusion is always true, $N_{\lambda} = [N_{\lambda} : N]N$.

(ii) $\implies$ (iii). Let $A'$ be a finite subset of $A$, and let $H = \sum_{\lambda \in A'} N_{\lambda}$. The fact that $H$ is multiplication follows from [1, Theorem 2.3]. As $N_{\lambda} = [N_{\lambda} : N]N$ for all $\lambda \in A'$, we have
\[ H = \sum_{\lambda \in A'} N_{\lambda} = \left( \sum_{\lambda \in A'} [N_{\lambda} : N] \right) N \subseteq [H : N]N. \]

(iii) $\implies$ (iv). Let $K$ be any submodule of $M$. For all $x \in K \cap N$, there exists a finite subset $\Lambda_x$ of $\Lambda$ such that $x \in \sum_{\lambda \in \Lambda_x} N_{\lambda} = N_x$. It follows that
\[ K \cap N = \sum_{x \in K \cap N} Rx \subseteq \sum_{x \in K \cap N} K \cap N_x. \]

$N_x$ is multiplication and $N_x = [N_x : N]N$. Hence
\[ K \cap N \subseteq \sum_{x \in K \cap N} [K : N_x]N_x \subseteq \sum_{x \in K \cap N} [K : N][N_x : N]N \subseteq [K : N]N \subseteq K \cap N \]
so that $K \cap N = [K : N]N$ and $N$ is multiplication.

(iv) $\implies$ (i) follows by Lemma 1.1.

This finishes the proof of the theorem. \qed

If $R$ is a ring and $N_{\lambda}(\lambda \in \Lambda)$ a collection of submodules of an $R$–module $M$ such that $\sum_{\lambda \in \Lambda} N_{\lambda}$ is multiplication, must all of $N_{\lambda}$ (or at least some of them) be multiplication modules? The answer is no, as the following example shows: Let $R$ be a ring with distinct maximal ideals $M_1$ and $M_2$. Then $R = M_1 + M_2$ is a multiplication ideal but neither $M_1$ nor $M_2$ is multiplication. (See for example [11, Example 38]).

We show that the answer is yes under an additional condition.
**Theorem 3.2.** Let $R$ be a ring and $N_\lambda (\lambda \in \Lambda )$ a collection of submodules of an $R$-module $M$ such that $N = \sum_{\lambda \in \Lambda } N_\lambda$ is multiplication. If, moreover, $N_\lambda \cap N_\mu$ is multiplication for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$, then all submodules $N_\lambda (\lambda \in \Lambda )$ are multiplication modules.

**Proof.** Let $A = \sum_{\lambda \in \Lambda } [N_\lambda : N]$. By Lemma 1.1, $N = AN$. Let $\lambda \in \Lambda$. Let $K \subseteq N_\lambda$. Clearly $[K : N_\lambda]N_\lambda \subseteq K$. For the reverse inclusion, let $y \in K$. Set

$$H = \{ r \in R | ry \in [K : N_\lambda]N_\lambda \}.$$  

Suppose that $H \neq R$. There exists a maximal ideal $P$ of $R$ such that $H \subseteq P$.

Case 1: $A \subseteq P$. Then $N = AN \subseteq PN \subseteq N$ so that $N = PN$. Next, $Ry = IN$ for some ideal $I$ of $R$ and hence

$$Ry = IN = IPN = P(IN) = Py.$$  

There exists $p \in P$ such that $(1-p)y = 0$ and hence $1-p \in H \subseteq P$, a contradiction.

Case 2: $A \not\subseteq P$. Then

$$[N_\lambda : N] + \sum_{\mu \neq \lambda} [N_\mu : N] \not\subseteq P.$$  

If $[N_\lambda : N] \not\subseteq P$, there exists $q \in P$ such that $(1-q)N \subseteq N_\lambda$. Now

$$K = [K : N]N \subseteq [K : N_\lambda]N,$$  

and hence

$$(1-q)y \in (1-q)K \subseteq [K : N_\lambda](1-q)N \subseteq [K : N_\lambda]N_\lambda.$$  

It follows that $1-q \in H \subseteq P$, also a contradiction. Finally if $\sum_{\mu \neq \lambda} [N_\mu : N] \not\subseteq P$, there exists $\mu \in \Lambda$ and $q' \in P$ such that $(1-q')N \subseteq N_\mu$. It follows that $(1-q')N_\lambda \subseteq N_\lambda \cap N_\mu$, and hence $(1-q')K \subseteq N_\lambda \cap N_\mu$. But $N_\lambda \cap N_\mu$ is multiplication. Thus

$$(1-q')K = J(N_\lambda \cap N_\mu)$$  

for some ideal $J$ of $R$. Next

$$JN_\lambda = (1-q')JN_\lambda \subseteq J(N_\lambda \cap N_\mu) = (1-q')K \subseteq K$$  

so that $(1-q')J \subseteq [K : N_\lambda]$, and hence

$$(1-q')^2y \in (1-q')^2K = (1-q')J(N_\lambda \cap N_\mu) \subseteq (1-q')JN_\lambda \subseteq [K : N_\lambda]N_\lambda.$$  

This gives a contradiction since $(1-q')^2 \in H \subseteq P$. Therefore $H = R$ and $x \in [K : N_\lambda]N_\lambda$ so that $K = [K : N_\lambda]N_\lambda$. This shows that $N_\lambda$ is a multiplication module. □

It is well-known that if $A$ and $B$ are $R$-modules, then $A \oplus B$ is projective if and only if $A$ and $B$ are projective, [10]. The same is not true for multiplication modules. for example, $R$ is a multiplication $R$-module but $R \oplus R$ is not. We apply Theorem 3.2 to give an alternative proof to [8, Theorem 2.2] characterizing modules which are direct sums of multiplication modules.

If $N_\lambda (\lambda \in \Lambda )$ is a non-empty collection of $R$-modules and $\lambda \in \Lambda$, we let $
abla \lambda = \bigoplus_{\mu \neq \lambda} N_\mu$ and $N = \bigoplus_{\lambda \in \Lambda} N_\lambda$. Note $N = N_\lambda \oplus \nabla \lambda$ for all $\lambda \in \Lambda$. We identify the $N_\lambda$ with their natural injections in $N$. 


Theorem 3.3. Let $R$ be a ring and $N_\lambda (\lambda \in \Lambda)$ a collection of $R$–modules. Then $N = \bigoplus_{\lambda \in \Lambda} N_\lambda$ is multiplication if and only if:

(i) $N_\lambda$ is multiplication for all $\lambda \in \Lambda$.

(ii) For each $\lambda \in \Lambda$, there exists an ideal $A_\lambda$ of $R$ such that $A_\lambda N_\lambda = N_\lambda$ and $A_\lambda \widehat{N_\lambda} = 0$.

**Proof.** Suppose first that $N$ is multiplication. Then $0 = N_\lambda \cap \widehat{N_\lambda} \supseteq N_\lambda \cap N_\mu \supseteq 0$, so that $N_\lambda \cap N_\mu = 0$ which is a multiplication module for all $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$. From Theorem 3.2 it follows that $N_\lambda$ is multiplication for all $\lambda \in \Lambda$. Let $\lambda \in \Lambda$. Then $N_\lambda = A_\lambda N = A_\lambda N_\lambda + A_\lambda \widehat{N_\lambda}$ for some ideal $A_\lambda$ of $R$. Hence $A_\lambda \widehat{N_\lambda} \subseteq N_\lambda \cap \widehat{N_\lambda} = 0$ so that $N_\lambda = A_\lambda N_\lambda$ and $A_\lambda \widehat{N_\lambda} = 0$. Conversely, suppose the modules $N_\lambda (\lambda \in \Lambda)$ satisfy (i) and (ii). Then

$$A_\lambda \subseteq \text{ann}(\widehat{N_\lambda}) \subseteq [N_\lambda : \widehat{N_\lambda}] = [N_\lambda : N],$$

and hence

$$N_\lambda \subseteq A_\lambda N \subseteq [N_\lambda : N]N \subseteq N_\lambda,$$

so that $N_\lambda = [N_\lambda : N]N$ for all $\lambda \in \Lambda$.

Put $A = \sum_{\lambda \in \Lambda} [N_\lambda : N]$. Then $N = AN$, and hence $N_\lambda = [N_\lambda : N]N = [N_\lambda : N]AN = A\widehat{N_\lambda}$ for all $\lambda \in \Lambda$. But $N_\lambda$ is multiplication for all $\lambda \in \Lambda$. Thus, by Lemma 1.1, $R = A + \text{ann}(x)$ for all $x \in N_\lambda (\lambda \in \Lambda)$. Let $n \in N$. There exist a finite subset $A'$ of $\Lambda$ and elements $x_\lambda \in N_\lambda (\lambda \in A')$ such that $n = \sum_{\lambda \in A'} x_\lambda$. It follows that

$$R = A + \bigoplus_{\lambda \in A'} \text{ann}(x_\lambda) = A + \text{ann} \left( \sum_{\lambda \in A'} R x_\lambda \right) \subseteq A + \text{ann}(n)$$

so that $R = A + \text{ann}(n)$. By Lemma 1.1, $N$ is multiplication. This concludes the proof of the theorem. \[\square\]

The next result should be compared with [8, Corollary 2.3].

**Corollary 3.4.** Let $R$ be a ring $N_\lambda (\lambda \in \Lambda)$ a collection of finitely generated $R$–modules. Then $N = \bigoplus_{\lambda \in \Lambda} N_\lambda$ is multiplication if and only if

(i) $N_\lambda$ is multiplication for all $\lambda \in \Lambda$.

(ii) $\text{ann}(N_\lambda) + \text{ann}(\widehat{N_\lambda}) = R$ for all $\lambda \in \Lambda$.

**Proof.** Suppose that $N$ is multiplication. Then (i) follows by Theorem 3.3. Let $\lambda \in \Lambda$. As $N_\lambda \cap \widehat{N_\lambda} = 0$, we infer from [1, Theorem 2.1] that

$$R = \text{ann} (N_\lambda \cap \widehat{N_\lambda}) = \text{ann}(N_\lambda) + \text{ann} (\widehat{N_\lambda}) + \text{ann}(x)$$

for all $x \in N$ and in particular for all $x \in N_\lambda$. As $N_\lambda$ is finitely generated, $R = \text{ann}(N_\lambda) + \text{ann}(\widehat{N_\lambda}) + \bigcap_{x \in N_\lambda} \text{ann}(x)$, and (ii) follows. Conversely, suppose the modules $N_\lambda (\lambda \in \Lambda)$ satisfy (i) and (ii). Let $\lambda \in \Lambda$. Then

$$R \subseteq \text{ann}(N_\lambda) + [N_\lambda : \widehat{N_\lambda}] = \text{ann}(N_\lambda) + [N_\lambda : N] \subseteq \text{ann}(N_\lambda) + \sum_{\mu \in \Lambda} [N_\lambda : N] \subseteq R$$

so that $R = \text{ann}(N_\lambda) + \sum_{\mu \in \Lambda} [N_\lambda : N]$ and by [18, Theorem 2 Corollary 2], $N$ is multiplication. \[\square\]
We close with the following property of the direct sum of a finite set of multiplication modules.

**Corollary 3.5.** Let $R$ be a ring and $N_i(1 \leq i \leq n)$ a finite collection of $R$-modules. Then $N = \oplus N_i$ is multiplication if and only if:

(i) $N_i$ is multiplication for all $i \in \{1, \ldots, n\}$.

(ii) $\text{ann}(N_i) + \text{ann}(N_j) + \text{ann}(a) = R$ for all $a \in N_i + N_j$ and all $i < j$.

**Proof.** The sufficiency is immediate by Theorem 3.3 and Corollary 3.4. Conversely, suppose the modules $N_i$ satisfy (i) and (ii). Let $i < j$. Then $[N_i : N_j] + [N_j : N_i] + \text{ann}(a) = R$ and hence $Ra = ([N_i : N_j] + [N_j : N_i])Ra$ for all $a \in N_i + N_j$. Summing over all $a \in N_i$, we get

$$N_i = [N_i : N_j]N_i + [N_i : N_j]N_j = [N_i : N_j]N_i + N_i \cap N_j,$$

so that

$$N_i = [N_i : N_j + N_j](N_i + N_j).$$

Similarly,

$$N_j = [N_j : N_i + N_j](N_i + N_j).$$

Let $K$ be a submodule of $N_i + N_j$. It follows from Lemma 1.1 that

$$K \cap (N_i + N_j) = K \cap N_i + K \cap N_j = [K : N_i]N_i + [K : N_j]N_j$$

$$= [K : N_i][N_i : N_i + N_j](N_i + N_j)$$

$$+ [K : N_j][N_j : N_i + N_j](N_i + N_j)$$

$$\subseteq [K : N_i + N_j](N_i + N_j) \subseteq K \cap (N_i + N_j),$$

so that

$$[K : N_i + N_j](N_i + N_j) = K \cap (N_i + N_j),$$

and hence $N_i + N_j$ is multiplication. It follows from [1, Theorem 2.3] that $N$ is multiplication, and the proof is complete. \(\square\)

In particular the case when $\Lambda$ is finite in Corollary 3.4 (or the modules $N_i$ are finitely generated in Corollary 3.5) is of interest. If $N_i(1 \leq i \leq n)$ is a finite collection of finitely generated $R$-modules, then $N = \oplus N_i$ is multiplication if and only if all $N_i$ are multiplication and $\text{ann}(N_i) + \text{ann}(N_j) = R$ for all $i < j$. In this case, $\text{ann}(N)$ is multiplication if and only if $\text{ann}(N_i)$ are all multiplication, see [18, Theorem 8(ii)] and [1, Theorem 2.3]. Also, using [1, Corollary 2.4], it is easily verified that if $\text{ann}(N_i)$ are finitely generated, then so too is $\text{ann}(N)$.

**References**


Majid M. Ali
Department of Mathematics
Sultan Qaboos University
Muscat
OMAN
mali@squ.edu.om

David J. Smith
Department of Mathematics
University of Auckland
Auckland
NEW ZEALAND
smith@math.auckland.ac.nz