SOME NEW EXPLICIT EVALUATIONS OF RAMANUJAN’S CUBIC CONTINUED FRACTION

CHANDRASHEKHAR ADIGA, K.R. VASUKI AND M.S. MAHADEVA NAIKA

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Abstract. In his first and ‘lost’ notebook, Ramanujan recorded several values of the Rogers - Ramanujan continued fraction and cubic continued fraction. In this paper we establish several new evaluations of Ramanujan’s cubic continued fraction using some modular equations.

1. Introduction

Ramanujan’s cubic continued fraction $G(q)$ is defined by

$$G(q) := \frac{q^{1/3} q + q^2 q^2 + q^4 \cdots}{1 + 1 + 1 + 1 + \cdots}, \quad |q| < 1.$$  

H.H. Chan [5] has proved several elegant theorems for $G(q)$, many of which are analogues of well-known properties satisfied by the Rogers - Ramanujan Continued Fraction:

$$R(q) := \frac{q^{1/3} q^2 q^3 \cdots}{1 + 1 + 1 + 1 + \cdots}, \quad |q| < 1.$$  

It is generally quite difficult to evaluate $G(-e^{-\pi \sqrt{n}})$ and $G(e^{-\pi \sqrt{n}})$ for specific values of $n$. Recently B.C. Berndt et al. [4] have proved general formulas for $G(-e^{-\pi \sqrt{n}})$ and $G(e^{-\pi \sqrt{n}})$ in terms of Ramanujan–Weber class invariants $G_n$ and $g_n$ which are defined by

$$G_n := 2^{-4} q^{-\frac{1}{24}} (-q; q^2)_\infty$$

and

$$g_n := 2^{-4} q^{-\frac{1}{24}} (q; q^2)_\infty, \quad q = e^{-\pi \sqrt{n}},$$

where $n$ is a positive rational number. Here $(a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n), \ |q| < 1$. Using these general formulas, Berndt et al. [4] have evaluated $G(-e^{-\pi \sqrt{n}})$ for $n = 1, 5, 13, 37$ and $G(e^{-\pi \sqrt{n}})$ for $n = 2, 10, 22, 58$. Other values of $G(q)$ can also be computed from the following reciprocity theorems proved by Chan [5], [6, Section 6.3]:

$$[1 - 2G(-e^{-\pi \alpha})][1 - 2G(-e^{-\pi \beta})] = 3$$  \hspace{1cm} (1.1)

and

$$[1 + G(-e^{-\sqrt{2}\pi \alpha})][1 + G(-e^{-\sqrt{2}\pi \beta})] = 3/2, \hspace{1cm} (1.2)$$

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where $\alpha, \beta > 0$ and $\alpha \beta = 1$. K.G. Ramanathan [7] has proved that

$$G(e^{-\pi \sqrt{10}}) = \frac{\sqrt{9 + 3\sqrt{6} - \sqrt{7 + 3\sqrt{6}}}}{(1 + \sqrt{5})\sqrt{6 + \sqrt{5}}}$$

which was given by Ramanujan on page 366 of his ‘lost’ notebook [9]. But Ramanathans proof depends upon Kronecker’s limit formula.

The main purpose of this paper is to determine some new values of $G(q)$ and $H(q) := -G(-q)$. For our evaluations we use some modular equations and transformation formulas stated by Ramanujan in his notebooks [8, pp. 199, 241, 324].

2. Main Theorems

**Theorem 2.1.** We have

$$H(e^{-\frac{5\pi}{\sqrt{3}}}) = \frac{1}{12} \left(t_1^\frac{1}{4} + t_2^\frac{1}{4}\right),$$

where

$$t_1 = 432 \left(5 + 2\sqrt{5}\right)$$

and

$$t_2 = -864 \left(2 + \sqrt{5}\right).$$

**Proof.** To prove this theorem we shall employ the following transformation formula [1, Entry 27(iv) of Chapter 16, p. 39]: If $\alpha, \beta > 0$ then

$$e^{-\frac{\alpha x}{\sqrt{3}}} \sqrt{\alpha} f(e^{-\alpha}) = e^{-\frac{\beta x}{\sqrt{3}}} \sqrt{\beta} f(e^{-\beta}).$$

Here

$$f(-q) = (q; q)_\infty, \quad |q| < 1.$$  

Let

$$P = \frac{f(q)}{q^{\frac{1}{12}} f(q^3)} \quad \text{and} \quad Q = \frac{f(q^5)}{q^{\frac{5}{12}} f(q^{15})}. \quad (2.3)$$

Then

$$(PQ)^2 + \frac{9}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 - \left(\frac{P}{Q}\right)^3 + 5. \quad (2.4)$$

This appears as equation 62.2 of Chapter 25 [3, p. 221]. Let $q = e^{-\pi/\sqrt{3}}$. Then putting $\alpha = \pi/\sqrt{3}, \beta = \sqrt{3}\pi$ in (2.2), we obtain

$$P = 3^{\frac{1}{4}}. \quad (2.5)$$

Using (2.5) in (2.4) and then setting $Q = i3^{1/4}T$, we obtain the equation

$$T^6 + 3iT^5 + 5iT^3 + 3iT + 1 = \left(T - \left[-\frac{i + i\sqrt{5}}{2}\right]\right)^3 \left(T - \left[-\frac{i - i\sqrt{5}}{2}\right]\right)^3 = 0.$$ 

Thus

$$T = \frac{-i \pm i\sqrt{5}}{2}.$$
If we took \( T = -\frac{i + \sqrt{5}}{2} \), we would find that \( Q = i^{3/4}T < 0 \). But clearly \( Q > 0 \), and so we deduce that
\[
Q = 3^{\frac{1}{4}} \frac{1 + \sqrt{5}}{2}.
\]

Replacing \( q \) by \(-q\) in Entry 1(iv) of Chapter 20 of Ramanujan's second notebook [8, p. 241], [2, p. 345], we obtain
\[
\left(27 - \frac{f^{12}(q)}{qf^{12}(q^3)}\right)\frac{1}{3} = \frac{-1}{4H(q)} + 4H^2(q).
\]

Setting \( q = e^{-5\pi/\sqrt{3}} \) in (2.7) and using (2.6), we find that
\[
4t^3 - \left[27 - 27\left(\frac{1 + \sqrt{5}}{2}\right)^{12}\right]t - 1 = 0,
\]
where
\[
t = H\left(e^{-\frac{5\pi}{\sqrt{3}}}\right).
\]

Solving this cubic equation, we find that
\[
\frac{1}{12}\left(t_1^\frac{1}{3} + t_2^\frac{1}{3}\right), \frac{1}{12}\left(\omega t_1^\frac{1}{3} + \omega^2 t_2^\frac{1}{3}\right) \text{ and } \frac{1}{12}\left(\omega^2 t_1^\frac{1}{3} + \omega t_2^\frac{1}{3}\right)
\]
are the roots. Since, the above cubic equation has at most one positive real root and \( t = H(e^{-5\pi/\sqrt{3}}) > 0 \), we conclude that \( t = \frac{1}{12}(t_1^{1/3} + t_2^{1/3}) \). □

**Theorem 2.2.** We have
\[
G(e^{-2\pi}) = \frac{1}{2} - \frac{\sqrt{3}}{4} \left[1 + \sqrt{3} - \sqrt{2\sqrt{3}}\right].
\]

**Proof.** To prove this theorem we use the following transformation formula [1, Entry 27(iii) of Chapter 16, p.39]: If \( \alpha, \beta > 0 \), then
\[
e^{-\frac{\alpha}{12}} \sqrt[6]{\alpha} f(-e^{-2\alpha}) = e^{-\frac{\beta}{12}} \sqrt[6]{\beta} f(-e^{-2\beta}).
\]

Setting \( \alpha = \pi/3, \beta = 3\pi \) in (2.9), we find that
\[
\frac{f(-e^{-\frac{2\pi}{3}})}{e^{-\frac{2\pi}{3}} f(-e^{-6\pi})} = \sqrt{3}.
\]

From Entry 1(iv) of Chapter 20 of Ramanujan's second notebook [8, p. 241], [2, p. 345], we have
\[
3 + \frac{f^3(-q^{\frac{1}{3}})}{q^{\frac{1}{3}} f^3(-q^3)} = \frac{1}{G(q)} + 4G^2(q).
\]

This can be rewritten as
\[
4Z^3 - (3 + 3\sqrt{3})Z + 1 = 0,
\]
where
\[
Z = G(e^{-2\pi}).
\]

Setting \( Z = -\frac{\sqrt{3}}{2}x + \frac{1}{2} \), we obtain the equation
\[
x^3 - \sqrt{3}x^2 - \sqrt{3}x + 1 = 0.
\]
The standard method of solving such an equation \([10, \text{pp. 27–28}]\) then gives us

\[
x = -1 \quad \text{or} \quad x = \frac{(\sqrt{3} + 1) \pm \sqrt{2\sqrt{3}}}{2}.
\]

If we took \(x = -1\), we would find that \(Z > 1\). If we took \(x = \frac{(\sqrt{3} + 1) + \sqrt{2\sqrt{3}}}{2}\), we would find that \(Z < 0\). But clearly \(0 < Z < 1\), and so we deduce that

\[
x = \frac{(\sqrt{3} + 1) - \sqrt{2\sqrt{3}}}{2}.
\]

Hence, it follows that

\[
Z = \frac{1}{2} - \frac{\sqrt{3}}{4} \left[ 1 + \sqrt{3} - \sqrt{2\sqrt{3}} \right].
\]

\[\square\]

**Theorem 2.3.** We have

(a) \[H(e^{-\frac{\pi}{\sqrt{3}}}) = \frac{1}{\sqrt{4}}, \quad (2.12)\]

(b) \[H(e^{-5\sqrt{3}}) = \frac{1}{12} \left( t_1 + t_2 \right), \quad (2.13)\]

where \[t_1 = 432 \left[ 5 - 2\sqrt{5} \right]\]

and \[t_2 = 864 \left( \sqrt{5} - 2 \right),\]

(c) \[H(e^{-7\sqrt{3}}) = \frac{1}{12} \left( t_1 + t_2 \right), \quad (2.14)\]

where \[t_1 = 864 \left( 14 + 3\sqrt{21} \right)\]

and \[t_2 = -1296 \left( 9 + 2\sqrt{21} \right),\]

(d) \[H(e^{-9\sqrt{3}}) = \frac{1}{12} \left( t_1 + t_2 \right), \quad (2.15)\]

where \[t_1 = 864 \left( 14 - 3\sqrt{21} \right)\]

and \[t_2 = 1296 \left( 2\sqrt{21} - 9 \right),\]

(e) \[G(e^{-2\sqrt{3}}) = \frac{1}{\sqrt{2}}. \quad (2.16)\]
**Proof.** As the proof of this theorem is identical with the proof of Theorems 2.1 and 2.2, we just give only references of the required transformation formula, modular equation and set of values of $\alpha$, $\beta$ and $q$.

To prove (2.12) we employ the transformation formula (2.2) and the equation (2.7) with $\alpha = \pi/\sqrt{3}$, $\beta = \sqrt{3}\pi$ and $q = e^{-\pi/\sqrt{3}}$.

To prove (2.13) we employ the transformation formula (2.2) and the equation (2.7) with $\alpha = \pi/\sqrt{3}$, $\beta = \sqrt{3}\pi$ and $q = e^{-\pi/3\sqrt{3}}$.

To prove (2.14) we employ the transformation formula (2.2) and the modular equation [3, Entry 69 of Chapter 25] with $\alpha = \pi/\sqrt{7}$, $\beta = \sqrt{7}\pi$ and $q = e^{-\pi/\sqrt{7}}$, and then we use (2.7) with $q = e^{-\pi/\sqrt{3}}$.

To prove (2.15) we employ the transformation formula (2.2) and the modular equation [3, Entry 69 of Chapter 25] with $\alpha = \pi/\sqrt{3}$, $\beta = \sqrt{3}\pi$ and $q = e^{-\pi/\sqrt{3}}$. Then we use (2.7) with $q = e^{-\pi/\sqrt{3}}$.

To prove (2.16) we employ the transformation formula (2.9) with $\alpha = \pi/\sqrt{3}$, $\beta = \sqrt{3}\pi$ and the equation (2.7) with $q = e^{-2\pi/\sqrt{3}}$. $\square$

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**References**


Chandrashekhar Adiga  
Department of Studies in Mathematics  
University of Mysore  
Manasagangotri  
Mysore 570006  
INDIA  
c_adiga@hotmail.com

K.R. Vasuki  
Department of Studies in Mathematics  
University of Mysore  
Manasagangotri  
Mysore 570006  
INDIA  
avsuki_k@hotmail.com

M.S. Mahadeva Naika  
Department of Mathematics  
Maharanis Science College for Women  
J.L.B. Road  
Mysore 570005  
INDIA  
msmnaika@rediffmail.com