A NONABSOLUTE INTEGRAL ON MEASURE SPACES THAT INCLUDES THE DAVIES–MC SHANE INTEGRAL

NG Wee Leng

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Abstract. We show that a Henstock–type integral on measure spaces endowed with metric topologies has the property that its absolute integrability is equivalent to the Davies integrability, the Davis–McShane integrability, the Lebesgue integrability and a McShane–type integrability.

In [1], Henstock constructed a division space from an arbitrary non-atomic measure space with a locally compact Hausdorff topology that is compatible with the measure, and defined the Davies–McShane integral. However, the integral defined is absolute in the sense that if \( f \) is integrable, so is \( |f| \).

We then showed in [4] that actually a nonabsolute integral, which we call the H-integral, can be defined on measure spaces and pointed out that the H-integral includes the Davis and the Davis–McShane integrals defined in [1]. In this paper, we shall prove that in fact, for measurable functions, the absolute H-integrability, the M-integrability, the Davies integrability, the Davies–McShane integrability and the Lebesgue integrability are all equivalent. The M-integral is a McShane-type integral which we shall define later. What we shall present here is a special case of the results we have proved in [3].

1. Preliminaries

Let \((X, d)\) be a metric space with topology \(T\) induced by the metric \(d\) on \(X\) and let \((X, \Omega, \iota)\) be a measure space such that \(T \subset \Omega\). The measure \(\iota\) is assumed to be non-negative and countably additive. Furthermore, the following condition will be assumed throughout this paper.

\((\ast)\): For every measurable set \(W \in \Omega\) and every \(\varepsilon > 0\), there exist an open set \(U\) and a closed set \(Y\) such that \(Y \subset W \subset U\) and \(\iota(U \setminus Y) < \varepsilon\).

Let \(T_1\) be the set of all open balls. A set of the form \(\{y \in X : d(x, y) < r\}\), where \(x \in X\) and \(r > 0\), denoted by \(B(x, r)\), is called an open ball. We shall also call its closure a closed ball. Throughout this paper we shall assume that \(\iota(B) > 0\) and \(\iota(B) = \iota(\overline{B})\) for all \(B \in T_1\), where \(\overline{B}\) denotes, as usual, the closure of \(B\).

Consider the following sets:

\[ \mathcal{H}_1 = \{ \overline{B_1 \setminus B_2} : B_1, B_2 \in T_1 \text{ where } B_1 \notin B_2 \text{ and } B_2 \notin B_1 \} , \]

\[ \mathcal{H}_2 = \left\{ \bigcap_{i \in \Lambda} X_i \neq \emptyset : X_i \in \mathcal{H}_1 \text{ and } \Lambda \text{ is a finite index set} \right\} . \]

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More precisely, members of $\mathcal{H}_1$ are either closed balls or scalloped balls. A typical member of $\mathcal{H}_2$ is a finite intersection of a combination of closed balls or scalloped balls.

We shall call members of $\mathcal{H}_2$ generalised intervals or simply intervals where there is no ambiguity. Note that intervals are relatively compact, though not necessarily closed or compact. Also note that $\iota(I) = \iota(\bar{I})$ for each interval $I$.

**Example 1.1.** Let $X$ be the real line and $\mathcal{T}$ the family of all open sets. Then $\mathcal{T}_1$ is the set of all open intervals $(a, b)$. It is easy to see that $\mathcal{H}_2$ is the set of all intervals of the form $(a, b)$, $(a, b]$, $[a, b)$ or $[a, b]$. In other words, generalised intervals in this case are the usual bounded intervals. Notice that taking the difference of two bounded intervals such that one does not include the other ensures that we obtain a connected interval rather than two disjoint bounded intervals.

**Example 1.2.** Let $X$ be the two-dimensional Euclidean space $\mathbb{R}^2$. The metrics $d_1$ and $d_2$ in $X$ are given by

$$d_1(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\},$$

$$d_2(x, y) = \left[ (x_1 - y_1)^2 + (x_2 - y_2)^2 \right]^{\frac{1}{2}},$$

for each $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $X$. It is well-known that the $d_1$-open balls are squares without the boundaries, and the $d_2$-open balls are open circular discs. It is easy to see that when the metric $d_1$ is used, a generalised interval looks like a polygon with edges each being either vertical or horizontal, and each edge is not necessarily included. When the metric $d_2$ is used instead, a generalised interval is a simply connected domain in the plane with piecewise circular edges, and each arc may or may not be included.

We next present the necessary and standard terminology in defining a Henstock-type integral.

Let $E$ be a finite union of (possibly just one) mutually disjoint intervals and call it an elementary set. Throughout this paper, we shall let an elementary set $E$ with finite measure, that is $\lambda(E) < +\infty$, be fixed and define integrability on $E$.

A set $\{(I_i, x_i) : i = 1, 2, \ldots, n\}$ of interval-point pairs is called a partial division of $E$ if $I_1, I_2, \ldots, I_n$ are mutually disjoint subintervals of $E$ such that $E \setminus \bigcup_{i=1}^{n} I_i$ is either empty or an elementary subset of $E$, and for each $i$, we have $x_i \in \overline{I_i}$. We call $x_i$ the associated point of $I_i$. A division of $E$ is a partial division $\{(I_i, x_i) : i = 1, 2, \ldots, n\}$ such that the union of $I_i$ is $E$.

A partial division $D^*$ of $E$ refines or is a refinement of another partial division $D$ of $E$ if for each $(I, x) \in D^*$, we have $I \subset J$ for some $(J, y) \in D$.

Let $\delta : \overline{E} \to \mathbb{R}^+$ be a positive function. We call $\delta$ a gauge on $E$. Note that we need to consider gauges defined on $\overline{E}$ and not just $E$, because for each interval-point pair $(I, x)$ in a partial division, the associated point $x$ comes from $\overline{I}$ and not just $I$.

Let a gauge $\delta$ on $E$ be given. An interval-point pair $(I, x)$ is $\delta$-fine if $I \subset B(x, \delta(x))$. A partial division $\{(I_i, x_i) : i = 1, 2, \ldots, n\}$ of $E$ is $\delta$-fine if $(I_i, x_i)$ is $\delta$-fine for each $i = 1, 2, \ldots, n$. Since divisions themselves are partial divisions, the $\delta$-fine divisions of $E$ are similarly defined. A gauge $\delta_1$ is said to be finer
than a gauge \( \delta_2 \) on \( E \) if for every \( x \in \overline{E} \) we have \( \delta_1(x) \leq \delta_2(x) \). The existence of \( \delta \)-fine divisions has been proved in [4].

We shall now define the \( H \)-integral and the \( M \)-integral. For brevity and where there is no ambiguity, \( D = \{(I, x)\} \) shall denote a finite collection of interval-point pairs \((I, x)\), and the corresponding Riemann sum shall be denoted by \((D) \sum f(x)i(I)\).

All functions \( f \) considered in this paper are real-valued point functions defined on \( E \).

**Definition 1.3.** A function \( f \) is said to be \( H \)-integrable on \( E \) to a real number \( A \) if for every \( \varepsilon > 0 \), there exists a gauge \( \delta \) on \( E \) such that for every \( \delta \)-fine division \( D = \{(I, x)\} \) of \( E \), we have

\[
\left| (D) \sum f(x)i(I) - A \right| < \varepsilon.
\]

We write \((H) \int_E f = A\). The \( H \)-integrability of \( f \) on any elementary subset of \( E \) is similarly defined.

It can be proved that the \( H \)-integral is uniquely determined, and closed under addition, scalar multiplication, monotone convergence and controlled convergence. Furthermore, the Cauchy criterion of integrability and Henstock's lemma also hold [3]. We have also proved in [4] that the \( H \)-integral is a nonabsolute one.

We next introduce the \( M \)-integral which is a McShane-type integral. Given a gauge \( \delta \) on \( E \), a set \( D = \{(I_i, x_i) : i = 1, 2, \ldots, n\} \) is called a \( \delta \)-fine McShane partial division of \( E \) if \( I_i \) are mutually disjoint subintervals of \( E \) such that \( I_i \subset B(x_i, \delta(x_i)) \) for each \( i = 1, 2, \ldots, n \) and where \( x_i \) is in \( E \), though not necessarily in \( \overline{I_i} \). Again, if further the union of \( I_i \) is \( E \), we call \( D \) a \( \delta \)-fine McShane division of \( E \). Obviously, a \( \delta \)-fine partial division of \( E \) is a \( \delta \)-fine McShane partial division of \( E \) but not conversely.

If (1.1) in Definition 1.3 holds for every \( \delta \)-fine McShane division \( D \) of \( E \), we say that \( f \) is \( M \)-integrable on \( E \) to the value \( A \) and we write \((M) \int_E f = A\). The \( M \)-integrability of \( f \) on any elementary subset of \( E \) is similarly defined. Obviously, a function which is \( M \)-integrable on \( E \) is \( H \)-integrable on \( E \). Also note that \( M \)-integrable functions are measurable and so are \( H \)-integrable functions [3].

Given a function \( f \) which is \( H \)-integrable on \( E \), the primitive \( F \) of \( f \) on \( E \) is given by

\[
F(E_0) = (H) \int_{E_0} f
\]

for each elementary subset \( E_0 \) of \( E \). Note that \( F \) is an elementary-set function which is finitely additive over elementary subsets of \( E \) (see [3]). Finite additivity here is defined in the standard manner. The primitive of a \( M \)-integrable function is similarly defined.

**2. The Equivalence of the \( M \)-Integrability and the Absolute \( H \)-Integrability**

This section aims to prove that a function is \( M \)-integrable on \( E \) if and only if it is absolutely \( H \)-integrable on \( E \). A function \( f \) is said to be absolutely \( H \)-integrable on \( E \) if \( f \) and its absolute value \(|f|\) are both \( H \)-integrable on \( E \).
The desired result will be proved in a few theorems. To this end, we need the following definition.

**Definition 2.1.** An elementary-set function $F$ is said to be AC on $E$ if for every $\varepsilon > 0$, there exists $\eta > 0$ such that for every partial division $D = \{(I, x)\}$ of $E$ satisfying the condition $(D) \sum \iota(I) < \eta$, we have

$$(D) \sum |F(I)| < \varepsilon.$$ 

The first theorem we will prove indicates the relevance of the above concept.

**Theorem 2.2.** If a function $f$ is $M$-integrable on $E$, then its primitive $F$ is AC on $E$.

**Proof.** Since $f$ is $M$-integrable on $E$, for every $\varepsilon > 0$, there is a gauge $\delta$ on $E$ such that for every $\delta$-fine McShane division $D = \{(I, x)\}$ of $E$,

$$(D) \sum |F(I) - f(x)\iota(I)| < \frac{\varepsilon}{2}.$$ 

Let $D_0$ be one such division and let

$$M = \max \{|f(x)| : (I, x) \in D_0\}.$$ 

Choose $\eta > 0$ such that $2M\eta < \varepsilon$. For every partial division $D$ of $E$ with $(D) \sum \iota(I) < \eta$, we partition $D$ into subintervals belonging to those in $D_0$ and denote by $D_1$ the new collection of intervals with associated points provided by $D_0$. Then we have

$$(D) \sum |F(I)| \leq (D_1) \sum |F(I) - f(x)\iota(I)| + (D_1) \sum |f(x)|\iota(I)$$

$$< \frac{\varepsilon}{2} + M\eta$$

$$< \varepsilon.$$ 

Hence $F$ is AC on $E$. \qed

The next theorem gives a sufficient condition for a $H$-integrable function to be absolutely $H$-integrable.

**Theorem 2.3.** If a function $f$ is $H$-integrable on $E$ and is bounded on $E$, then $f$ is absolutely $H$-integrable on $E$.

**Proof.** Since $f$ is $H$-integrable on $E$, for every $\varepsilon > 0$, there is a gauge $\delta_1$ such that for every $\delta_1$-fine division $D = \{(I, x)\}$ of $E$, we have

$$(D) \sum |f(x)\iota(I) - F(I)| < \varepsilon$$

where $F$ is the primitive of $f$. Next, for every gauge $\delta$ on $E$ we put

$$A_\delta = \sup_{D_\delta} (D_\delta) \sum |F(I)|$$

where the supremum is taken over all $\delta$-fine divisions $D_\delta = \{(I, x)\}$ of $E$ and let

$$A = \inf_\delta A_\delta.$$
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where the infimum is taken over all gauges $\delta$ on $E$. Note that since $f$ is bounded, $A$ exists and is finite. Then there exists a $\delta_1$–fine division $D_0$ of $E$ such that for every $\delta_1$–fine division $D$ of $E$ which refines $D_0$, we have

$$|A - (D) \sum |F(I)|| < \varepsilon.$$ 

Now, since $X$ is a metric space, we can choose a gauge $\delta_2$ finer than $\delta_1$ such that every $\delta_2$–fine division $D$ of $E$ can be partitioned into a refinement $D'$ of $D_0$. For convenience, we still write $D$ for $D'$. So take any $\delta_2$–fine division $D = \{(J, \xi)\}$ of $E$ and we obtain

$$|(D) \sum |f(\xi)||\iota(J) - A| \leq |(D) \sum |f(\xi)||\iota(J) - (D) \sum |F(J)||$$

$$+ |(D) \sum |F(J)| - A| < 2\varepsilon.$$ 

This completes the proof. 

In the above proof the boundedness of $f$ is used only to verify that $A$ is finite. Hence Theorem 2.3 holds true as long as $A$ is finite, where the latter is true, for example, when the primitive $F$ of $f$ is $AC$ on $E$.

The next theorem has been proved in [5] for the real line. As the proof can be easily generalised to our setting, we shall state the theorem without proof here.

**Theorem 2.4.** If a function $f$ is $H$–integrable on $E$, then $f_1$ is also $H$–integrable on $E$ where $f_1(x) = f(x)$ when $\alpha < f(x) \leq \beta$, where $\alpha < \beta$, and 0 otherwise.

The above theorem suggests a way of extending the domains of $H$–integrability and $M$–integrability to measurable sets. Given a function $f$ on $\overline{E}$ and for every measurable subset $W$ of $\overline{E}$, the function $f_W$ is given by $f_W(x) = f(x)$ if $x \in W$ and 0 otherwise. This leads to the following definition.

**Definition 2.5.** Let $W$ be a measurable subset of $\overline{E}$. A function $f$ is said to be $H$–integrable on $W$ to a real number $A$ if $f_W$ is $H$–integrable on $E$ to the number $A$ and we write $(H) \int_W f = A$. If $f_W$ is absolutely $H$–integrable on $E$, then we say that $f$ is absolutely $H$–integrable on $W$.

If $f$ is absolutely $H$–integrable on $W$, then it is also meaningful to write $(H) \int_W |f|$. Note that by definition, if $f$ is $H$–integrable on $W$ to the value $A$, then

$$(H) \int_W f = (H) \int_E f_W = A.$$ 

If $F$ is the primitive of $f$ on $E$, we can also write $F(W) = A$. The $M$–integrability on a measurable subset of $\overline{E}$ is similarly defined.

**Theorem 2.6.** If a function $f$ is $H$–integrable on $E$ and its primitive $F$ is $AC$ on $E$, then for every $\varepsilon > 0$, there is $\eta > 0$ such that for any measurable subset $Y$ of $\overline{E}$ on which $f$ is absolutely $H$–integrable with $\iota(Y) < \eta$, we have

$$(H) \int_Y |f| < \varepsilon.$$
Proof. Let $\varepsilon > 0$ be given and let $\eta > 0$ be such that for every partial division $D = \{(I, x)\}$ of $E$ with $(D) \sum \iota(I) < \eta$, we have

$$(D) \sum |F(I)| < \frac{\varepsilon}{4}.$$ 

Let $Y$ be a measurable subset of $E$ on which $f$ is absolutely $H$-integrable with $\iota(Y) < \frac{\eta}{2}$. By Condition (*), we can choose an open set $U$ such that $Y \subset U$ and $\iota(U \setminus Y) < \frac{\eta}{2}$. We also choose a gauge $\delta$ on $E$ such that $x \in B(x, \delta(x)) \subset U$ if $x \in Y$ and that for every $\delta$-fine division $D = \{(I, x)\}$ of $E$, we have

$$(D) \sum_{x \in Y} |f(x)| \iota(I) < (H) \int_Y |f| < \frac{\varepsilon}{2}. \quad (2.1)$$

Now let $D = \{(J, \xi)\}$ be a $\delta$-fine division of $E$ and note that

$$(D) \sum_{\xi \in Y} \iota(J) \leq \iota(U) = \iota(U \setminus Y) + \iota(Y) < \eta.$$ 

Therefore $(D) \sum_{\xi \in Y} |F(J)| < \frac{\varepsilon}{4}$. In view of Henstock's lemma we may also assume that

$$(D) \sum_{\xi \in Y} |f(\xi)| \iota(J) - F(J)| < \frac{\varepsilon}{4}.$$ 

It follows that $(D) \sum_{\xi \in Y} |f(\xi)| \iota(J) < \frac{\varepsilon}{2}$. Finally, by applying (2.1), we obtain $(H) \int_Y |f| < \varepsilon$ as desired. □

The above theorem is essential in proving the next result.

**Theorem 2.7.** If a function $f$ is $H$-integrable on $E$ and its primitive $F$ is $AC$ on $E$, then $f$ is $M$-integrable on $E$.

Proof. Let $\varepsilon > 0$ be given and let $\eta > 0$ be such that the condition as described in Theorem 2.6 is satisfied. Write

$$X_k = \{x \in E : (k - 1)\varepsilon < f(x) \leq k\varepsilon\}$$

for $k = 0, \pm 1, \pm 2, \ldots$. Next, for each $k$ take an open set $U_k \supset X_k$ such that

$$\iota(U_k \setminus X_k) < \frac{\eta}{(|k| + 1)2^{|k|+2}}.$$ 

By virtue of Theorem 2.4, the function $f$ is absolutely $H$-integrable on each $X_k$. Let

$$Y = \bigcup_{k=-\infty}^{\infty} (U_k \setminus X_k).$$

Then $\iota(Y) < \eta$ and so

$$(H) \int_Y |f| < \varepsilon.$$ 

Now define a gauge $\delta$ on $E$ such that $B(\xi, \delta(\xi)) \subset U_k$ when $\xi \in X_k$ for $k = 0, \pm 1, \pm 2, \ldots$. For convenience, we write $X_\xi = X_k$ when $\xi \in X_k$ for some
and assume $\eta \leq 1$. In view of the absolute $H$–integrability of $f$ on each $X_k$, for every $\delta$–fine McShane division $D = \{(J, \xi)\}$ of $E$, we obtain

$$
(D) \sum f(\xi) \mu(J) - F(E) \leq (D) \sum |f(\xi) - f|
$$

$$
\leq (D) \sum |f(\xi) - f| + (D) \sum |f(\xi)| + (D) \sum |f|
$$

$$
\leq \varepsilon \cdot \mu(E) + 2\varepsilon,
$$

and the result follows.

**Theorem 2.8.** If a function $f$ is absolutely $H$–integrable on $E$, then its primitive $F$ is $AC$ on $E$.

**Proof.** For a fixed integer $N$, let $f^N(x) = f(x)$ when $|f(x)| \leq N$ and 0 otherwise. Since $f$ is $H$–integrable on $E$, so is $f^N$ by Theorem 2.4. It is easy to see that the primitive $F^N$ of the bounded function $f^N$ is $AC$ on $E$. So the primitive $G^N$ of $|f^N|$ is $AC$ on $E$. That is, for every $\varepsilon > 0$, there is $\eta > 0$ such that for every partial division $D$ of $E$ with $(D) \sum \mu(I) < \eta$, we have

$$(D) \sum |G^N(I)| < \varepsilon.$$

By virtue of the monotone convergence theorem, there is a positive integer $N$ such that

$$(D) \sum |F(I)| \leq (D) \sum \int_I |f| \leq \left[ (D) \sum \int_I |f| - (D) \sum \int_I |f^N| \right] + (D) \sum |G^N(I)| < 2\varepsilon.$$

Hence $F$ is $AC$ on $E$. □

With Theorems 2.7 and 2.8, we therefore arrive at the following result.

**Theorem 2.9.** If a function $f$ is absolutely $H$–integrable on $E$, then it is $M$–integrable on $E$.

In view of the remark after Theorem 2.3, we can yield the following result by modifying the proof of Theorem 2.3 and by applying Theorem 2.2.

**Theorem 2.10.** If $f$ is $M$–integrable on $E$, then so is $|f|$.

Since a $M$–integrable function on $E$ is $H$–integrable on $E$, by Theorems 2.9 and 2.10, we have established the following result as desired.

**Theorem 2.11.** Let $Y$ be a measurable subset of $\overline{E}$. A function $f$ is $M$–integrable on $Y$ if and only if it is absolutely $H$–integrable on $Y$. 
3. The Final Result

The equivalence of the Davis, the Davis–McShane and the Lebesgue integrals for measurable functions has been proved in [1]. It therefore remains to show that the $M$–integral is equivalent to the Lebesgue integral.

Let $Y$ be a measurable subset of $E$. The Lebesgue integral of a function $f$ on $Y$, if exists, will be denoted by $(L) \int_Y f$.

It is well–known that the Lebesgue integral is absolutely continuous [2, Theorem 12.34] in the sense that for every $\varepsilon > 0$, there exists $\eta > 0$ such that for every measurable set $W \subset Y$ such that $\mu(W) < \eta$, we have $(L) \int_W |f| < \varepsilon$.

In the proof of the following theorem, we will use the fact that a measurable function bounded on a measurable set $Y$ is Lebesgue integrable on $Y$.

**Theorem 3.1.** Let $Y$ be a measurable subset of $E$. A function $f$ is $M$–integrable on $Y$ if and only if it is Lebesgue integrable on $Y$.

**Proof.** Suppose $f$ is $M$–integrable on $Y$. In view of Theorem 2.10, we may assume that $f$ is non–negative on $Y$. For each $n$, we let $Y_n$ be the subset of $Y$ such that $0 \leq f(x) \leq n$ for $x \in Y_n$ and let $f_n = f|_{Y_n}$. Note that $f_n \to f$ as $n \to \infty$ almost everywhere in $Y$. Now, since $f$ is $M$–integrable, so is $f_n$ by Theorems 2.4 and 2.11. Then since $M$–integrable functions are measurable, each $f_n$ is a bounded measurable function on $Y$. Consequently, each $f_n$ is Lebesgue integrable on $Y$. Finally, since $\{f_n\}$ is a monotone sequence of functions in $Y$ where $\lim_{n \to \infty} (L) \int_Y f_n = (M) \int_Y f < \infty$, by Levi’s Theorem [2, Theorem 12.22], the function $f$ is Lebesgue integrable on $Y$. Conversely, if $f$ is Lebesgue integrable on $Y$, by the absolute continuity of the Lebesgue integral, a proof similar to that of Theorem 2.7 will show that $f$ is $M$–integrable on $Y$.

With the equivalence of the Davis-McShane and the Lebesgue integrals established in [1], the following result follows immediately from Theorems 2.11 and 3.1.

**Theorem 3.2.** Let $Y$ be a measurable subset of $E$. A function $f$ is Davies–McShane integrable on $Y$ if and only if it is absolutely $H$–integrable on $Y$.

We can now conclude that for measurable functions, the absolute $H$–integral, the $M$–integral, the Davies integral, the Davies–McShane integral and the Lebesgue integral are all equivalent. In particular, the $H$–integral includes the Davies–McShane integral defined in [1].

**References**


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Ng Wee Leng
Mathematics and Mathematics Education
National Institute of Education
Nanyang Technological University
1 Nanyang Walk
SINGAPORE 637616
wlng@nie.edu.sg