
1. Introduction

Let $X$ be a real or complex Banach space. The norm on $X$ and on the Banach algebra $B(X)$ of all bounded linear operators from $X$ into itself will be denoted by $\| \cdot \|$.

We recall ([1], [3]) that an operator-valued map $S : \mathbb{R}^+ \to B(X)$ is called a $C_0$-quasisemigroup on $X$ if it has the following properties:

$$S(0, t_0) = I$$

(the identity operator on $X$) for every $t_0 \geq 0$;

$$S(t, s + t_0)S(s, t_0) = S(t + s, t_0)$$

for all $(t, s, t_0) \in \mathbb{R}^3_+$;

$$\lim_{t \to 0} \| S(t, t_0)x_0 - x_0 \| = 0$$

for all $(t_0, x_0) \in \mathbb{R}_+ \times X$;

there exists an increasing function $\omega : \mathbb{R}_+ \to \mathbb{R}^*_+$ such that:

$$\| S(t, t_0) \| \leq \omega(t) \quad \text{for all} \quad (t, t_0) \in \mathbb{R}^2_+.$$

Remark 1.1. If $S : \mathbb{R}^+ \to B(X)$ is a $C_0$-semigroup on $X$ then

$$\tilde{S} : \mathbb{R}^2_+ \to B(X), \quad \tilde{S}(t, t_0) \overset{d}{=} S(t)$$

is a $C_0$-quasisemigroup on $X$. 

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Definition 1.2. The $C_0$-quasisemigroup $S : \mathbb{R}_+^2 \to \mathcal{B}(X)$ is called uniform dichotomic (and we write u.d.) if there are $N \geq 1$ and a strongly continuous projection-valued function $P : \mathbb{R}_+ \to \mathcal{B}(X)$, such that

$$S(t, t_0)P(t_0) = P(t + t_0)S(t, t_0)$$

(d)

$$\|S(t + s, t_0)x_0\| \leq N \|S(t, t_0)x_0\|$$

(d)

$$\|S(s, t_0)y_0\| \leq N \|S(t + s, t_0)y_0\|$$

(d)

for all $(t, s, t_0) \in \mathbb{R}_+^3$, $x_0 \in \text{Im} P(t_0)$ and all $y_0 \in \text{Ker} P(t_0)$.

Remark 1.3. The $C_0$-quasisemigroup $S$ is u.d. if and only if there exists a projection valued function $P : \mathbb{R}_+ \to \mathcal{B}(X)$ with the property (d0) and such that the inequalities (d1) and (d2) hold for all $(t, s, t_0) \in \mathbb{R}_+^3$ with $t \geq 1$ and all $x_0 \in \text{Im} P(t_0)$ and $y_0 \in \text{Ker} P(t_0)$.

Proof. Indeed, if $t \in [0, 1]$, $(s,t_0) \in \mathbb{R}_+^2$ and $x_0 \in \text{Im} P(t_0)$ then

$$\|S(t + s, t_0)x_0\| \leq \|S(t, s + t_0)||S(s, t_0)x_0||$$

$$\leq \omega(1) \|S(s, t_0)x_0||$$

$$\leq N \omega(1) \|S(s, t_0)x_0||,$$

and for $y_0 \in \text{Ker} P(t_0)$ we have

$$\|S(s, t_0)y_0\| \leq N \|S(s + 1, t_0)y_0||$$

$$\leq N \|S(1 - t, t + s + t_0)||S(s + t, t_0)y_0||$$

$$\leq N \omega(1) \|S(t + s, t_0)y_0||.$$

Definition 1.4. The $C_0$-quasisemigroup $S : \mathbb{R}_+^2 \to \mathcal{B}(X)$ is called uniformly exponentially dichotomic (and we denote u.e.d.) if there are $N \geq 1$, $\nu > 0$ and a strongly continuous projection-valued function $P : \mathbb{R}_+ \to \mathcal{B}(X)$ such that

$$P(t + t_0)S(t, t_0) = S(t, t_0)P(t_0)$$

(d0)

$$e^{\nu t} \|S(t + s, t_0)x_0\| \leq N \|S(s, t_0)x_0||$$

(ed1)

$$e^{\nu s} \|S(t, t_0)y_0\| \leq N \|S(t + s, t_0)y_0||$$

(ed2)

for all $(t, s, t_0) \in \mathbb{R}_+^3$, $x_0 \in \text{Im} P(t_0)$ and all $y_0 \in \text{Ker} P(t_0)$.

Remark 1.5. It is obvious that if $S$ is u.e.d. then it is u.d. Similarly as in the proof of Remark 1.3 we can prove that $S$ is u.e.d. if and only if (d0), (ed1) and (ed2) are satisfied for all $(t, s, t_0) \in \mathbb{R}_+^3$ with $t \geq 1$ and all $x_0 \in \text{Im} P(t_0)$ and $y_0 \in \text{Ker} P(t_0)$. 
2. Auxiliary Results

We start with the following

**Lemma 2.1.** Let $f : \mathbb{R}_+ \to \mathbb{R}_+^*$ be a continuous function with the property that there are $a, b > 0$ such that:

$$af(s) \leq \int_s^\infty f(t) \, dt \leq bf(s) \quad \text{for all } s \geq 0.$$

Then

$$ae^{t/b}f(t + s) \leq be^{1/b}f(s)$$

for all $s \geq 0$ and all $t \geq 1$.

**Proof.** The function $F : \mathbb{R}_+ \to \mathbb{R}_+^*$ defined by:

$$F(t) = \int_t^\infty f(s) \, ds$$

is differentiable with

$$F(s) \leq bf(s) = -bF'(s) \quad \text{for all } s \geq 0.$$

Then

$$F(t + s) \leq F(s)e^{-t/b} \quad \text{for all } (t, s) \in \mathbb{R}_+^2.$$

Finally, we obtain that for all $s \geq 0$ and all $t \geq 1$ we have

$$af(t + s) \leq F(t + s - 1) \leq F(s)e^{-(t-1)/b} \leq be^{1/b}e^{-t/b}f(s)$$

for all $(t, s) \in \mathbb{R}_+^2$. \hfill \Box

**Lemma 2.2.** Let $g : \mathbb{R}_+ \to \mathbb{R}_+^*$ be a continuous function with the property that there are $a, b > 0$ such that:

$$g(s) \leq ag(t + s) \quad (i)$$

and

$$\int_0^t g(s) \, ds \leq bg(t) \quad (ii)$$

for all $(t, s) \in \mathbb{R}_+^2$. Then

$$e^{t/b}g(s) \leq abe^{1/b}g(t + s)$$

for all $s \geq 0$ and all $t \geq 1$.

**Proof.** The function $G : \mathbb{R}_+ \to \mathbb{R}_+^*$ defined by

$$G(t) = \int_0^t g(s) \, ds$$

is differentiable with

$$bG'(t) = bg(t) \geq G(t).$$
If \( t \geq 1 \) then
\[
abg(t+s) \geq aG(t+s) \geq aG(t+1)e^{\frac{t-1}{b}}
\]
\[
\geq aG(1)e^{\frac{t-1}{b}} = ae^{\frac{t-1}{b}} \int_s^{s+1} g(s+v) \, dv
\]
\[
\geq e^{\frac{t-1}{b}} g(s)
\]
for all \( s \geq 0 \).

**Lemma 2.3.** Let \( S \) be a \( C_0 \)-quasisemigroup with the property that there exists \( N > 0 \) such that
\[
\int_t^{\infty} |S(\tau,t_0)x_0| \, d\tau \leq N |S(t,t_0)x_0|
\]
for all \((t,t_0,x_0) \in \mathbb{R}_+^2 \times X\). Then there exists \( M > 0 \) such that
\[
|S(t+s,t_0)x_0| \leq M |S(s,t_0)x_0|
\]
for all \((t,s,t_0,x_0) \in \mathbb{R}_+^3 \times X\).

**Proof.** If \( t \geq 1 \) and if we denote
\[
\frac{1}{m} = \int_0^1 ds \omega(s),
\]
where \( \omega \) is given by condition (q4) from definition of the notion of \( C_0 \)-quasisemigroup, then
\[
|S(t+s,t_0)x_0| = \int_{t+s-1}^{t+s} |S(t+s,t_0)x_0| \, dv \\
\leq \int_s^{\infty} |S(v,t_0)x_0| \, dv \\
\leq N |S(s,t_0)x_0|.
\]
For \( t \in [0,1] \) we have
\[
|S(t+s,t_0)x_0| \leq |S(t,s+t_0)||S(s,t_0)x_0| \\
\leq \omega(1) |S(s,t_0)x_0|.
\]
Finally we obtain the desired inequality with
\[
M = mN + \omega(1).
\]

**Lemma 2.4.** Let \( S \) be a \( C_0 \)-quasisemigroup with the property that there exists \( N > 0 \) such that
\[
\int_t^{t+1} |S(s,t_0)x_0| \, ds \leq N |S(t,t_0)x_0|
\]
for all \((t,t_0,x_0) \in \mathbb{R}_+^2 \times X\). Then there exists \( M > 0 \) such that
\[
|S(s,t_0)x_0| \leq M |S(t+s,t_0)x_0|
\]
for all \((t,s,t_0,x_0) \in \mathbb{R}_+^3 \times X\) with \( s \geq 1 \).
Proof. If $s \geq 1$ and $m$ is defined as in the proof of Lemma 2.3 then

\[
\frac{\|S(s, t_0)x_0\|}{m} = \int_{s-1}^{s} \frac{\|S(s, t_0)x_0\|}{\omega(s-v)} \, dv
\]

\[
\leq \int_{s-1}^{s} \|S(v, t_0)x_0\| \, dv
\]

\[
\leq \int_{0}^{t+s} \|S(v, t_0)x_0\| \, dv
\]

\[
\leq N \|S(t + s, t_0)x_0\|
\]

and hence

\[
\|S(s, t_0)x_0\| \leq mN \|S(t + s, t_0)x_0\|
\]

for all $(t, s, t_0, x_0) \in \mathbb{R}^3_+ \times X$ with $s \geq 1$. \qed

A sufficient condition for uniform dichotomy is given by

**Theorem 2.5.** Let $S$ be a $C_0$–quasisemigroup with the property that there are a strongly continuous projection–valued function $P : \mathbb{R}^+ \to B(X)$ and a constant $N > 0$ such that

\[
\int_{t}^{\infty} \|S(\tau, t_0)x_0\| \, d\tau \leq N \|S(t, t_0)x_0\|; \quad (d_0)
\]

\[
\int_{0}^{t} \|S(s, t_0)y_0\| \, ds \leq N \|S(t, t_0)y_0\| \quad (d_2')
\]

\[
\|S(s, t_0)y_0\| \leq N \|S(s + 1, t_0)y_0\| \quad (d_3')
\]

for all $(t, s, t_0) \in \mathbb{R}^3_+$, $x_0 \in \text{Im } P(t_0)$ and all $y_0 \in \text{Ker } P(t_0)$. Then $S$ is uniformly dichotomic.

**Proof.** It results from Lemma 2.3, Lemma 2.4, Remark 1.3 and the observation that for $s \in [0, 1]$ we have from $(d_3')$ that

\[
\|S(s, t_0)y_0\| \leq N \|S(s + 1, t_0)y_0\|
\]

\[
\leq mN \|S(t + s, t_0)y_0\|
\]

for all $t \geq 1$, $t_0 \geq 0$ and $y_0 \in \text{Ker } P(t_0)$. \qed

3. The Main Result

In this section we give necessary and sufficient conditions for uniform exponential dichotomy of $C_0$–quasisemigroups of linear operators in Banach spaces.

**Theorem 3.1.** Let $S$ be a uniform dichotomic $C_0$–quasisemigroup. Then $S$ is uniformly exponentially dichotomic if and only if there exists $N \geq 1$ such that:

\[
\int_{t}^{\infty} \|S(\tau, t_0)x_0\| \, d\tau \leq N \|S(t, t_0)x_0\|; \quad (ed_1')
\]
and 
\[ \int_0^t \| S(s, t_0)y_0 \| \, ds \leq N \| S(t, t_0)y_0 \| ; \quad \text{(ed'}_2) \]
for all \((t, t_0) \in \mathbb{R}^2_+, x_0 \in \text{Im } P(t_0)\) and all \(y_0 \in \text{Ker } P(t_0)\).

**Proof. Necessity.** It is a simple verification.

**Sufficiency.** (1) If we consider the function 
\[ f: \mathbb{R}^+ \to \mathbb{R}^+, \quad f(t) = \| S(t, t_0)x_0 \| \]
where \(t_0 \in \mathbb{R}^+\) and \(x_0 \in \text{Im } P(t_0)\) then from (ed'1) it follows that
\[ \int_s^\infty f(t) \, dt \leq N f(t). \]
On the other hand,
\[ \frac{f(s + 1)}{m} = \int_0^1 \frac{\| S(s + 1, t_0)x_0 \|}{\omega(v)} \, dv \]
\[ \leq \int_s^{s+1} \frac{\| S(s + 1 - t, t + t_0) \| \| S(t, t_0)x_0 \|}{\omega(s + 1 - t)} \, dt \]
\[ \leq \int_s^\infty \| S(t, t_0)x_0 \| \, dt = F(s) \]
for all \(s \geq 0\), where \(m\) is defined as in the proof of the Lemma 2.3. We can apply Lemma 2.1 to the function \(f\) and we obtain that
\[ e^{\nu t} \| S(t + s, t_0)x_0 \| \leq N_1 \| S(s, t_0)x_0 \| \]
where \( \nu = \frac{1}{N} \) and \( N_1 = (m + N) Ne^{1/N} \)
for all \((t, s, t_0) \in \mathbb{R}^3_+\), with \(t \geq 1\) and all \(x_0 \in \text{Im } P(t_0)\).

(2) Consider the function 
\[ g: \mathbb{R}^+ \to \mathbb{R}^+, \quad g(t) = \| S(t, t_0)y_0 \| \]
where \(t_0 \in \mathbb{R}^+\) and \(y_0 \in \text{Ker } P(t_0)\). From the hypothesis it follows that there are \(M, N > 0\) such that
\[ g(s) \leq Mg(t + s) \quad \text{and} \quad \int_0^t g(s) \, ds \leq Ng(t). \]
for all \((t, s) \in \mathbb{R}^2_+\). By Lemma 2.2 it results that
\[ e^{\nu t} \| S(s, t_0)y_0 \| \leq N_1 \| S(t + s, t_0)y_0 \| \]
for all \((t, s, t_0) \in \mathbb{R}^3_+\), with \(t \geq 1\) and all \(y_0 \in \text{Ker } P(t_0)\). From Remark 1.5 we obtain that \(S\) is u.e.d. \(\square\)
Theorem 3.2. A $C_0$–quasisemigroup is uniformly exponentially dichotomic if and only if there are a strongly continuous projection-valued function $P : \mathbb{R}^+ \to \mathcal{B}(X)$ and a constant $N \geq 1$ such that

$$S(t, t_0)P(t_0) = P(t + t_0)S(t, t_0); \quad (d_0)$$

$$\int_t^\infty \|S(\tau, t_0)x_0\| \, d\tau \leq N \|S(t, t_0)x_0\|; \quad (ed_1')$$

$$\int_0^t \|S(s, t_0)y_0\| \, ds \leq N \|S(t, t_0)y_0\|; \quad (ed_2')$$

$$\|S(s, t_0)y_0\| \leq N \|S(s + 1, t_0)y_0\| \quad (ed_3')$$

for all $(t, s, t_0) \in \mathbb{R}^3_+, x_0 \in \text{Im} P(t_0)$ and all $y_0 \in \text{Ker} P(t_0)$.

Proof. It results from Theorems 3.1 and 2.5. \hfill \Box

Remark 3.3. In the particular case when $S$ is a $C_0$–semigroup from Theorem 3.2 we obtain a characterization of uniform exponential stability of $C_0$–semigroups. This characterization is a well–known result obtained by Datko in [2] and Pazy in [5].

In the particular case when $S$ is a $C_0$–semigroup Theorem 3.2 gives a characterization of uniform exponential dichotomy of $C_0$–semigroups. This result has been obtained by Preda and Megan in [6].

Thus the above theorem can be considered as a generalization of the results obtained by Datko [2] and Pazy [5], Preda and Megan [6]. We remark that our proofs of characterizations of the dichotomy properties are not generalizations of Datko and Pazy’s proof (resp. Preda and Megan’s proof).

For the particular case of uniform exponential stability property we obtain some results from [4].

A discrete variant of the Theorem 3.1 is

Theorem 3.4. Let $S$ be a uniform dichotomic $C_0$–quasisemigroup. Then $S$ is uniformly exponentially dichotomic if and only if there exists $N \geq 1$ such that

$$\sum_{k=n}^\infty \|S(k, t_0)x_0\| \leq N \|S(n, t_0)x_0\|; \quad (ed_1'')$$

and

$$\sum_{k=0}^n \|S(k, t_0)y_0\| \leq N \|S(n, t_0)y_0\|; \quad (ed_2'')$$

for all $(n, t_0) \in \mathbb{N} \times \mathbb{R}^+, x_0 \in \text{Im} P(t_0)$ and all $y_0 \in \text{Ker} P(t_0)$.

Proof. Necessity. It is immediate by verification.
**Sufficiency.** Let \((t, t_0) \in \mathbb{R}^2_+\) and \(n \in \mathbb{N}\) such that \(n \leq t < n + 1\). Then

\[
\int_t^\infty \|S(\tau, t_0)x_0\| \, d\tau \leq \int_t^{n+1} \|S(\tau, t_0)x_0\| \, d\tau \\
+ \sum_{k=n+1}^{\infty} \int_k^{k+1} \|S(\tau - k, k + t_0)\| \|S(k, t_0)x_0\| \, d\tau \\
\leq \omega(1) \left( \|S(t, t_0)x_0\| + \sum_{k=n+1}^{\infty} \|S(k, t_0)x_0\| \right) \\
\leq \omega(1) \left( \|S(t, t_0)x_0\| + N \|S(n + 1, t_0)x_0\| \right) \\
\leq N_1 \|S(t, t_0)x_0\|
\]

and

\[
\int_0^t \|S(s, t_0)y_0\| \, ds \leq \int_0^{n+2} \|S(s, t_0)y_0\| \, ds \\
+ \sum_{k=0}^{n+1} \int_k^{k+1} \|S(s - k, k + t_0)\| \|S(k, t_0)y_0\| \, ds \\
\leq \omega(1) \left( \sum_{k=0}^{n+1} \|S(k, t_0)y_0\| \right) \leq N\omega(1) \left( \|S(n + 1, t_0)y_0\| \right) \\
\leq N\omega^2(1) \left( \|S(t, t_0)y_0\| \right) \leq N_1 \|S(t, t_0)y_0\|
\]

for all \(x_0 \in \text{Im } P(t_0)\) and all \(y_0 \in \text{Ker } P(t_0)\), where

\[N_1 = \omega(1)(1 + N\omega(1)).\]

From Theorem 3.1 we obtain that \(S\) is u.e.d.

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