A NOTE ON THE STABILITY PROPERTIES OF THE EULER METHODS FOR SOLVING STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper we study the linear stability properties of the Euler methods for stochastic differential equations (SDEs) applied with constant stepsize. We extend the T-stability defined by Saito and Mitsui [10] from weak solutions to strong solutions. The extended definition of T-stability is equivalent to the asymptotic stability property of a numerical method applied to the linear test equation, which is also studied by Higham [5]. As the T-stability property has a strong relationship with the number of steps, T(A)-stability is defined in this paper. We prove that the implicit Euler method is T-stable for certain values of the linear test problem and give the T(A)-stability regions of the Euler methods. The numerical results indicate that the definition of T(A)-stability is meaningful.

1. Introduction

In this paper we consider the stability properties of three Euler methods for the stochastic differential equations (SDEs) driven by a d-dimensional Wiener process, interpreted in Itô form,

\[ dy = f(t,y)dt + \sum_{j=1}^{d} g_j(t,y)dW_j(t), \quad y(0) = y_0, \quad 0 \leq t \leq T, \]

(1)

where \( f(t,y) \) is called the drift coefficient, \( g_j(t,y) \) is called the diffusion coefficient, \( W_j(t) \) is the standard Wiener process whose increment \( \Delta W_j(t) = W_j(t+\Delta t) - W_j(t) \) is a Gaussian random variable \( \mathcal{N}(0, \Delta t) \).

Recently much work has been done in developing numerical schemes for SDEs [1] [2] [6], and these schemes can be divided into three categories.

(1) Explicit numerical schemes in which both the drift coefficient and the diffusion coefficients are explicit.

(2) Semi-implicit numerical schemes in which the drift coefficient is implicit but the diffusion coefficients are explicit.

(3) Implicit numerical schemes in which both the drift coefficient and the diffusion coefficients are implicit.

For a given time discretization of the interval \([t_0, T]\) with equidistant step size, let

\[ h = \frac{T}{N}, \quad t_n = t + nh, \quad (n = 1, 2, \ldots, N), \]

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where $y_n$ stands for the numerical solution of SDE (1) at step point $t_n$. Here we have the following three Euler methods for (1) which will be analysed in this paper.

(1) The explicit Euler method (the Euler–Maruyama method)

$$y_{n+1} = y_n + f(t_n, y_n)h + \sum_{j=1}^{d} g_j(t_n, y_n)\Delta W_{n_j}.$$  

(2) The semi-implicit Euler method

$$y_{n+1} = y_n + f(t_{n+1}, y_{n+1})h + \sum_{j=1}^{d} g_j(t_n, y_n)\Delta W_{n_j}.$$  

(3) The implicit Euler method

$$y_{n+1} = y_n + f(t_{n+1}, y_{n+1})h + \sum_{j=1}^{d} g_j(t_{n+1}, y_{n+1})\Delta W_{n_j}.$$  

Here the increment of the Wiener process $\Delta W_n$ is represented by $\Delta W_{n_j} = \sqrt{h} I_{n_j}$, where $I_{n_j}$ is the $n$th realization of $I_j$, the standard Gaussian random variable $N(0, 1)$.

Note, it can be shown that the explicit Euler method and the semi-implicit Euler method converge to the Itô solution of (1). While the implicit Euler method will converge as $h \to 0$ to the so-called backward solution which is based on evaluating the stochastic function at the righthand end point $t_{n+1}$ of the subinterval $[t_n, t_{n+1}]$.

The following linear test equation driven by the $d$-dimensional Wiener process

$$dy = aydt + \sum_{j=1}^{d} b_jydW_j(t)$$

is used to consider the stability properties of the Euler methods.

We can use a simple method to determine the stability property of the Euler methods (2) (3) (4). Denote $W^* = \sum_{j=1}^{d} b_jI_j$, so that $W^*$ is a Gaussian random variable $W^* \sim N(0, \sqrt{\sum_{j=1}^{d} b_j^2})$. It can be seen that the numerical stability properties of the Euler methods applied to the linear test equation (5) are equivalent to those applied to the following linear equation

$$dy = aydt + \sqrt{\sum_{j=1}^{d} b_j^2}ydt.$$

So we just consider the stability properties of the Euler methods applied to SDEs driven by one Wiener process and the linear test equation (in Itô form) is

$$dy = aydt + bydW(t), \quad y(0) = y_0.$$  

For the strong solution up to now, nearly all of the derived numerical methods are explicit or semi-implicit methods in nature. The difficulty with implicit methods is their stability properties. Applying the implicit Euler method to the linear Itô
test equation (7), Kloeden and Platen [6] point out that the implicit Euler method is not suitable as an approximation because the update values given by

$$y_{n+1} = \frac{1}{1 - ah - b\Delta W_n} y_n$$

may not exist or may be very large depending on $W_n$. Furthermore, it can be shown that this numerical approximation converges to

$$y(t) = e^{(a + \frac{1}{2} b^2)t + b(W(t))}y_0,$$

which is the backward solution of the linear test equation. It seems that implicit methods involving unbounded random variable are not useful, except perhaps in special cases such as linear equations with a strongly attracting drift and a very weak diffusion coefficient.

The stability properties of the numerical methods for SDEs are much more complicated to analyse than those for ordinary differential equations (ODEs) because of the uncertainty of the random variable $\Delta W_n$. Apply a one–step method for ODEs to the linear test equation

$$\frac{dy}{dt} = \lambda y,$$

which is stable for $\Re(\lambda) < 0$ since the fundamental solution is $y(t) = e^{\lambda t}$, then the method is represented by

$$y_{n+1} = R(\lambda h)y_n.$$ 

It is numerically stable iff

$$|R(\lambda h)| < 1 \iff \lim_{n \to \infty} R(\lambda h)^n = 0. \quad (8)$$

This means that $|R(\lambda h)|^N$ is very small for a relatively large $N$. At least for any $N$, $|R(\lambda h)|^N$ is less than 1.

Applying a one–step method for SDEs to the linear test equation (7) gives

$$y_{n+1} = R(h, a, b, \Delta W_n)y_n. \quad (9)$$

Compared with the stability analysis of the numerical methods for an ODE, we have the following problems in the stability analysis of the numerical methods for a SDE.

1. Corresponding to the first condition in (8), we cannot say the following

$$|R(h, a, b, \Delta W_n)| < 1 \quad (10)$$

holds deterministically as $\Delta W_n$ is a Gaussian random variable. For the given $h$, $a$ and $b$, we only have

$$P\{|R(h, a, b, \Delta W_n)| < 1\} = p.$$ 

The probability method can be used to check if the condition (10) holds. This leads to the Mean–Square stability definition [11].

2. Corresponding to the second condition in (8), we can determine whether

$$\lim_{n \to \infty} \prod_{i=1}^{n} R(h, a, b, \Delta W_i) = 0.$$
But this does not guarantee \( \prod_{i=1}^{n} R(h, a, b, \Delta W_n) \) is small (or at least less than 1) for a given positive number \( n \). This is important in the practical computations when the number of the steps is not very large.

(3) In solving a nonlinear ODE, we can regard \( \lambda \) as an eigenvalue of the Jacobian of \( f(t, y) \) with respect to \( y \) if \( f(t, y) \) does not change much in a subinterval. The stability properties of a numerical method for linear test ODE can be extended to that for nonlinear ODE. For a given \( h \), we can expect good numerical performance for solving the nonlinear ODE if the method for ODE is stable for given \( h \) and \( \lambda \).

Considering numerical methods for a SDE, there are additional difficulties between the stability analysis of the numerical methods for nonlinear equations than for linear equations. For a given \( \Delta W_n \), \( |R(h, a, b, \Delta W_n)| \) may be large, but for the linear test SDE, the equation (9) is solvable with a very large error if \( |R(h, a, b, \Delta W_n)| \) is large. This error may be corrected in the following steps. But for a nonlinear SDE, a nonlinear equation for \( y_{n+1} \) may be difficult to solve if \( |R(h, a, b, \Delta W_n)| \) is large. The numerical procedure may fail at this step unless other techniques are used.

In this paper, we discuss the stability definitions for the numerical methods for SDEs and discuss the stability properties of the Euler methods. In Section 2, we first review the existing stability definitions and the results. We give a method to calculate the \( T \)-value in the \( T \)-stability definition for the strong solution of SDEs and give the definition of \( T(A) \)-stability which is an extension of the \( T \)-stability definition. In Section 3, the stability properties of the Euler methods are discussed. We point out that the implicit Euler method is asymptotically stable for some \( a \), \( b \) and \( h \). To our knowledge, this is the first positive result for the stability property of the implicit Euler method. We give the stability regions of the Euler methods in this section. The numerical results in Section 4 give evidence that the \( T(A) \)-stability definition is meaningful. Conclusions are given at the end of the paper.

2. Definitions of Stochastic Stability

In this section we consider the stability properties of the numerical methods for SDE (1) applied to the linear test equation (7). The exact Itô solution of the linear test equation is

\[
y(t) = e^{(a - \frac{1}{2} b^2)(t - t_0) + bW(t)} y(t_0),
\]

and the problem is said to be mean-square stable [11] if

\[
\lim_{t \to \infty} E[y^2(t)] = 0 \iff R(a) + \frac{1}{2} b^2 < 0, \tag{11}
\]


\[
\lim_{t \to \infty} y(t) = 0 \iff R \left\{ a - \frac{1}{2} b^2 \right\} < 0.
\]

Higham [5] studies the asymptotic stability property of a numerical method applied to the linear test equation. The following definition and theorem can be found in [5].
Definition 2.1. The sequence \( \{y_n\} \) in (9) is said to be asymptotically stable if
\[
\lim_{n \to \infty} |y_n| = 0, \quad \text{with probability 1,}
\]
and for a particular choice of \( a, b \) and \( h \), we will say the numerical method is asymptotically stable if it produces an asymptotically stable sequence.

Theorem 2.2. Given a sequence of real-valued, non-negative, independent and identically distributed random variables \( \{Z_n\} \), consider the sequence of random variables \( \{Y_n\} \) defined by
\[
Y_n = \left( \prod_{i=0}^{n-1} Z_i \right) Y_0,
\]
where \( Y_0 \geq 0 \) and where \( Y_0 \neq 0 \) with probability 1. Suppose that the random variables \( \log(Z_i) \) are square-integrable. Then
\[
\lim_{n \to \infty} Y_n = 0, \quad \text{with probability 1} \iff E(\log(Z_i)) < 0 \quad \forall i.
\]

Let \( Z_i = |R(h, a, b, \sqrt{h}I_i)| \), then a numerical method is asymptotically stable if
\[
E\left(\log\left(|R(h, a, b, \sqrt{h}I)|\right)\right) = \int_{-\infty}^{\infty} \log \left(|R(h, a, b, \sqrt{h}x)|\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx < 0. \quad (12)
\]

Saito and Mitsui [11] consider the stability definition for the linear test equation (7). Let a stochastic scheme (applied over one step) be presented by (9) and
\[
R_2(h, a, b) = (|R(h, a, b, \Delta W_n)|^2),
\]
where \( R_2(h, a, b) \) is called the mean-square stability function of the numerical scheme. The following definition can be found in [11].

Definition 2.3. The numerical scheme is said to be mean-square (MS) stable for those value of \( h, a \) and \( b \) satisfying
\[
|R_2(h, a, b)| < 1,
\]
and the MS-stability region is given by
\[
S_2 = \{(h, a, b) : |R_2(h, a, b)| < 1\}.
\]

Another definition is that of T-stability defined by Saito and Mitsui. To analyse T-stability, a numerical scheme applied to (7) is averaged over \( n + 1 \) time steps to obtain an averaged one-step difference equation of the form
\[
\bar{y}_{n+1} = R_T(h, a, b)\bar{y}_n.
\]
\( R_T(h, a, b) \) could be obtained by
\[
R_T(h, a, b) = \sqrt{\prod_{i=1}^{n} R(h, a, b, \Delta W_i)}
\]
and is called the T-stability function and the following definition can be found in [10].
Definition 2.4. The numerical scheme is said to be $T$-stable if

$$\lim_{n \to \infty} |\tilde{y}_n| = 0 \iff |R_T(h,a,b)| < 1,$$

and the region of $T$-stability is

$$S_T = \{h,a,b : |R_T(h,a,b)| < 1\}.$$

Saito and Mitsui give the $T$-stability function of the Euler-Maruyama scheme (2) for the weak solution. The increment $\Delta W_n$ is known to be well approximated as $\Delta W_n = U_n \sqrt{h}$. As an example, $U_n$ is the two-point random variable

$$P(U_n = \pm 1) = \frac{1}{2}. \quad (13)$$

It is enough to take two steps for averaging. The $T$-stability function is then

$$R_T(h,a,b) = (1 + ha + b\sqrt{h})^2 (1 + ha - b\sqrt{h})^2.$$

Using the distribution (13) to consider the MS-stability, the mean-square stability function is

$$R_2(h,a,b) = \frac{1}{2} (1 + ha + b\sqrt{h})^2 + \frac{1}{2} (1 + ha - b\sqrt{h})^2.$$

The MS-stability corresponds to the arithmetic average but $T$-stability corresponds to the geometric average. The definition of $T$-stability is therefore weaker than that of MS-stability [10].

When considering $T$-stability for strong solutions, we give a method to compute the $T$-stability region when the driven process is a Wiener process. For a given number $l$, discretize the interval $[-M_1, M_1]$ as

$$-M_1 = x_0 < x_1 < \ldots < x_l = M_1. \quad (14)$$

Let

$$p_i = \frac{1}{\sqrt{2\pi}} \int_{x_{i-1}}^{x_i} e^{-\frac{x^2}{2}} dx, \quad u_i \in (x_{i-1}, x_i), \quad i = 1,2,\ldots,l.$$

Now $I_1$, the standard Gaussian random variable could be approximated by a discrete random variable $U$ with distribution

$$
\begin{array}{c|cccc}
   & u_1 & u_2 & \ldots & u_l \\
\hline
   p_i & p_1 & p_2 & \ldots & p_l 
\end{array}
$$

Assume $m$ random numbers are generated. There are $\lceil mp_i \rceil$ numbers in the interval $[x_{i-1}, x_i]$ and it is assumed these numbers are all equal to $u_i$. We can compute

$$T^m = \prod_{i=1}^{l} |R(h,a,b,\sqrt{h}u_i)|^{\lceil mp_i \rceil}$$

or the average

$$T = T(h,a,b) = \prod_{i=1}^{l} |R(h,a,b,\sqrt{h}u_i)|^{p_i}, \quad (15)$$

and say the numerical scheme is $T$-stable if $T < 1$. 
A NOTE ON THE STABILITY PROPERTIES

When the integral in (12) exists, log(T) can be regarded as an approximated value of the integral as

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \log(|R(h, a, b, \sqrt{h} x)|) e^{-\frac{x^2}{2}} \, dx \approx \frac{1}{\sqrt{2\pi}} \sum_{i=1}^{m} \int_{x_{i-1}}^{x_i} \log(|R(h, a, b, \sqrt{h} x)|) e^{-\frac{x^2}{2}} \, dx
\]

\[
\approx \sum_{i=1}^{m} \log \left( |R(h, a, b, \sqrt{h} u_i)| \right) p_i = \log(T).
\]

Then we can say that the definition of asymptotic stability is equivalent to that of T-stability. If the integral in (12) does not exist, the T-value is meaningless. In this case the T-value is heavily dependent on the choice of \( u_i \). Special attention should be paid to the case where the integral in (12) exists and is an improper integral with a singularity. In this case the subintervals in the neighbourhood of the infinite point should be very small because the T-value is sensitive to the choice of \( u_i \) if the subintervals are not very small.

From the T-stability definition, the numerical scheme is stable if

\[
T < 1 \iff \lim_{n \to \infty} T^n = 0.
\]

This definition is based on a very large sample size. In practical computations, the convergence rate of \( T^n \to 0 \) is needed when the number of timesteps is not very large. If \( T \) is close to 1, \( T^k \) is not a very small value, where \( k \) is a possibly small number of timesteps in practical computation. In addition, as \( R(h, a, b, \Delta W_n) \) is a random variable, its absolute value may be less than 1 or larger than 1. The probability for \( |R(h, a, b, \Delta W_n)| > 1 \) is not small when \( T \) is close to 1. We may obtain unstable results as

\[
\prod_{i=1}^{k} R(h, a, b, \sqrt{h} v_i) > 1,
\]

where \( v_i \) is the generated random number. Thus in order to guarantee the convergence rate of \( T^n \to 0 \), a stricter condition \( T < A < 1 \) is needed.

Based on the discussion about the value of \( T \), we have the following definition.

**Definition 2.5.** The numerical scheme is said to be \( T(A) \)-stable if

\[
T(h, a, b) < A,
\]

where \( 0 < A < 1 \).

3. The Stability Region of the Euler Methods

Applying the Euler methods (2) (3) (4) to the linear test equation (7) with equidistant step size \( h > 0 \), we have

1. The explicit Euler method:

\[
y_{n+1} = (1 + p + q I_n) y_n.
\]

2. The semi-implicit Euler method:

\[
y_{n+1} = \frac{1 + q I_n}{1 - p} y_n.
\]
(3) The implicit Euler method:

\[ y_{n+1} = \frac{1}{1 - p - qI_n} y_n, \]

where \( p = ah, \ q = b\sqrt{h}, \ \Delta W_n = \sqrt{h}I_n, \ I_n \) is the \( n \)th realization of \( I \), the standard Gaussian random variable \( N(0, 1) \).

For the explicit Euler method, the MS–stability function and the MS–stability region satisfy

\[ E(R_{21}) = (1 + p)^2 + q^2, \quad \mathcal{R}_{21} = \{(p, q) : \ |q| < \sqrt{-2p - p^2}, \ p < 0\}. \]

The MS–stability function and the MS–stability region of the semi–implicit Euler method satisfy

\[ E(R_{22}) = \frac{1 + q^2}{(1 - p)^2}, \quad \mathcal{R}_{22} = \{(p, q) : \ |q| < \sqrt{p^2 - 2p}, \ p < 0\}. \]

The MS–stability region of the implicit Euler method does not exist.

Now we consider the T-stability properties of the Euler methods. For the explicit and the semi–implicit Euler methods, the stability regions are given in Figures 1 and 2 with \( A = 0.6, \ 0.7, \ 0.8, \ 0.9, \ 1.0 \), respectively.

Now for the implicit Euler method, it is easy to verify the integral

\[ \int_{-\infty}^{\infty} \log^2 \left( \frac{1}{1 - p - qx} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx \]

exists for any \( p \) and \( q \) as

\[ \lim_{x \to \frac{1}{q} - p} (1 - p - qx)^\frac{1}{2} \log^2 \left( \frac{1}{1 - p - qx} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = 0. \]

Thus the implicit Euler method is asymptotically stable for some \( p \) and \( q \) if the following inequality holds for these \( p \) and \( q \)

\[ \int_{-\infty}^{\infty} \log \left( \frac{1}{1 - p - qx} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx < 0 \quad \text{or} \quad T = \prod_{i=1}^{l} \left| \frac{1}{1 - p - qu_i} \right|^{p_i} < 1. \quad (16) \]

It should be noticed that we only study the asymptotical stability property of the implicit Euler method here, namely consider the conditions under which the product

\[ \prod_{i=1}^{n} \frac{1}{1 - p - qu_i} \]

is convergent. Although this converges as \( h \to 0 \) to \( e^{(a + \frac{1}{2}b^2)t + bW(t)} y_0 \), so that, for example, when \( a = 0 \), as \( h \to 0 \) the implicit Euler method is unstable. It may happen that for large \( b \) and a given \( h \) the implicit Euler method is stable in that

\[ \left| \prod_{i=1}^{l} \frac{1}{1 - b\Delta W_i} - \prod_{i=1}^{l} (1 + b\Delta W_i) \right| = O \left( h^{\frac{1}{2}} \right) \]

and the implicit Euler method approximates the Itô solution (see Figure 4).

In this paper we choose \( M_1 = 8 \). For a given \( p \) and \( q \), special attention should be paid if \( \frac{1 - p}{q} \) is in \([-8, 8]\) when calculating the T-value using (16). Assuming
we should discretize \([\xi_{i+1}, a^+i]\) with smaller subintervals. With these very small intervals, the probability \(p_i\) is very small. The value

\[
\frac{1}{1 - p - qu_i}
\]

is very close to 1 although \(\frac{1}{1 - p - qu_i}\) is very large. In this way we can obtain accurate results.

In Figure 3 we give the T(A)–stability regions of the implicit Euler method. The area enclosed by the curve and the axis is the unstable region. As an example, we give the T-values of the Euler methods with \(p = -0.2\) and different \(q\) in Figure 4. The explicit and semi–implicit Euler method are stable when \(q\) is small in magnitude. Then the T-values are increased when the magnitude of \(q\) is increased. The stability property of the implicit Euler method is different. If \(p\) is small in magnitude, it is stable when \(q\) is small or large in magnitude. It is stable for any \(q\) when \(p\) is large in magnitude.

4. Numerical Results

In this section some numerical results are reported. The test equation is

\[
dy = -10ydt + bydW(t), \quad y(0) = 1.
\]
We repeat $M = 20$ batches and each batch contains $N = 500$ different simulations of sample paths of the Itô process and their Euler approximations corresponding to the same sample paths of the Wiener process. We use the mean and the confidence interval of the absolute error $[7]$ as the criteria, but only give the mean of the absolute errors because the radius of the confidence intervals are of the same order of those of the corresponding mean of the absolute errors.

Here we only consider the numerical results using $q$ in the T(A)–stability region. The maximum $q$ which is in the stability regions for $p = -1$ is listed in Table 1 for the explicit Euler method and the semi–implicit Euler method. For the implicit Euler method, we list the value $q$ in the upper boundary of the T(A)–stability regions for $p = -0.2$.

<table>
<thead>
<tr>
<th>Table 1. The maximum $q$ in the stability region with $p = -1$</th>
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<tbody>
<tr>
<td>$q$</td>
</tr>
<tr>
<td>-----</td>
</tr>
<tr>
<td>explicit method $(p = -1)$</td>
</tr>
<tr>
<td>semi–implicit method $(p = -1)$</td>
</tr>
<tr>
<td>implicit method $(p = -0.2)$</td>
</tr>
</tbody>
</table>

In Tables 2 and 3, we give numerical convergence results of the explicit and semi–implicit Euler methods with stepsize $h = 0.1$, $p = ah = -1$ to the Itô solution. These two methods have nearly the same order of accuracy if $q$ is in the same T(A)–stability region. From the results in the tables, $q$ should at least be in the T(0.7)–stability region when $n \leq 100$ and in the T(0.6)–stability region when $n \leq 40$ in order to obtain stable numerical results. They are unstable in the T(0.8)–stability region when $n \leq 100$. They are stable in the T(0.8)–stability region but are unstable in the T(0.9)–stability region when $200 \leq n \leq 1000$, and are stable in the T(0.9)–stability region but are unstable in the T(1)–stability region when $2000 \leq n \leq 10000$.

<table>
<thead>
<tr>
<th>Table 2. Means of accuracy of explicit Euler method</th>
</tr>
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<tbody>
<tr>
<td>$q$</td>
</tr>
<tr>
<td>-----</td>
</tr>
<tr>
<td>n = 20</td>
</tr>
<tr>
<td>n = 40</td>
</tr>
<tr>
<td>n = 60</td>
</tr>
<tr>
<td>n = 80</td>
</tr>
<tr>
<td>n = 10^2</td>
</tr>
</tbody>
</table>

The next test equation comes from the stochastic heat transfer equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \varepsilon u \xi(t), \quad 0 \leq x \leq 1, \quad t \geq 0$$

with the initial and boundary conditions

$$u(x, 0) = 1, \quad u(0, t) = 5, \quad \left. \frac{\partial u}{\partial t} \right|_{x=1} = 0,$$
TABLE 3. Means of accuracy of semi-implicit Euler method

<table>
<thead>
<tr>
<th>$q$</th>
<th>2.05</th>
<th>2.41</th>
<th>2.85</th>
<th>2.85</th>
<th>3.19</th>
<th>3.19</th>
<th>3.59</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 20$</td>
<td>0.2740</td>
<td>1.9428</td>
<td>51.682</td>
<td>59.022</td>
<td>3.9E+9</td>
<td>2.0E-38</td>
<td>3.5E+55</td>
</tr>
<tr>
<td>$n = 40$</td>
<td>0.0086</td>
<td>6.8509</td>
<td>284.52</td>
<td>5.0E-8</td>
<td>1.0E+10</td>
<td>3.6E-94</td>
<td>1.4E+91</td>
</tr>
<tr>
<td>$n = 60$</td>
<td>4.1E-5</td>
<td>0.3638</td>
<td>704.67</td>
<td>9.6E-15</td>
<td>1.8E+13</td>
<td>7.0E-202</td>
<td>2.3E+71</td>
</tr>
<tr>
<td>$n = 80$</td>
<td>3.2E-7</td>
<td>0.0297</td>
<td>7.3E+3</td>
<td>1.2E-27</td>
<td>5.3E+3</td>
<td>1.9E-272</td>
<td>5.7E+110</td>
</tr>
<tr>
<td>$n = 10^2$</td>
<td>3.9E-9</td>
<td>0.0017</td>
<td>831.46</td>
<td>2.0E-45</td>
<td>1.3E+4</td>
<td>0</td>
<td>1.2E+97</td>
</tr>
</tbody>
</table>

and $\xi(t)$ is the white noise process at time $t$.

For a given equidistant discretization

$$0 = x_0 < x_1 < \ldots < x_n = 1$$

with the step size $h_x = x_{i+1} - x_i$, using the central difference for the second order differential and the forward difference for the boundary condition, we have the following stochastic ordinary differential system

$$dU = QUdt + \varepsilon UdW(t) + U_0dt,$$

where

$$U_i = u(x_i, t), \quad i = 0, 1, \ldots, n, \quad U = (u_1, u_2, \ldots, u_{n-1}),$$

$$U_0 = \left(\frac{u_0}{h_x^2}, 0, \ldots, 0\right),$$

$$Q = \frac{1}{h_x^2} \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & 1 & -2 & 1 \\ & & & & 1 & -1 \end{pmatrix}.$$  

The exact solution of the equation (17) is

$$U(t) = \Phi(t)(U_0 + \int_0^t \Phi^{-1}(s)dsU_0),$$

with fundamental solution

$$\Phi(t) = \exp\left(\left(Q - \frac{1}{2}\varepsilon^2 E\right)t + \varepsilon EW(t)\right),$$

where $E$ is the identity matrix, $U_0$ is the initial condition.

The eigenvalues of the matrix $Q - \frac{1}{2}\varepsilon^2 E$ satisfy

$$\lambda(Q - \frac{1}{2}\varepsilon^2 E) = \lambda(Q) - \frac{1}{2}\varepsilon^2 < 0,$$

where $\lambda(A)$ is an eigenvalue of the matrix $A$. The solution of the equation (17) is stochastically asymptotically stable.

The formulae using the Euler methods to solve the equation (17) are listed below.

1. The explicit Euler method

$$U_{n+1} = ((1 + \varepsilon I_n \sqrt{h_t})E + Qh_t)U_n + U_0h_t.$$
(2) The semi-implicit Euler method

\[ U_{n+1} = (E - Qh_t)^{-1}[(1 + \varepsilon I_n \sqrt{h_t})U_n + \bar{U}_0h_t]. \]

(3) The implicit Euler method

\[ U_{n+1} = ((1 - \varepsilon I_n \sqrt{h_t})E - Qh_t)^{-1}(U_n + \bar{U}_0h_t). \]

For the explicit Euler method, the time step size \( h_t \) must satisfy the stability condition of the difference scheme

\[ \frac{h_t}{2h_x^2} < 1. \]

If this condition is not satisfied, we get a very oscillatory solution. Here \( h_x = 0.01, h_t = 0.00005 \) are used in the computation. Let

\[ p = \max (\lambda(Q))h_t, \quad q = \varepsilon \sqrt{h_t}. \]

The explicit Euler method is stable if \( p \) and \( q \) are in the MS-stability region of the explicit Euler method, namely \( \varepsilon < 2.23 \) when \( p = -1.25 \times 10^{-4}, q = 0.0158 \). In fact we can also get a stable solution if \( \varepsilon \) is a little larger than 2.23.

There is no restriction on the time step size \( h_t \) for the semi-implicit and the implicit Euler methods. The semi-implicit Euler method is stable if \( p \) and \( q \) are in the MS-stability region of the semi-implicit Euler method, namely \( \varepsilon < 2.30 \) when \( h_x = 0.01, h_t = 0.05, p = -0.125, q = 0.179 \). In fact we can obtain a stable solution if \( \varepsilon < 5 \).

The stability properties of the implicit Euler method are different from those of the other Euler methods. The implicit Euler method is stable if \( \varepsilon \) is small, for example \( \varepsilon < 0.8 \) when \( h_x = 0.01 \) and \( h_t = 0.05 \). If \( \varepsilon \) is larger than 0.8 but is not large, we normally get a solution with very high or very low temperature, \(|u| \gg 5\), at \( x = 1 \), which is an unstable solution. When \( \varepsilon \) is large, for example \( \varepsilon = 10 \), we may get an oscillatory solution with one time step and get a reasonable solution with another time step. In this case we regard that the solutions for large \( \varepsilon \) are unacceptable. Thus we can only obtain stable solutions with the implicit Euler method if \( \varepsilon \) is small.

5. Conclusion

In this paper the stability properties of the Euler methods are discussed. We give a formula to calculate \( T \)-values of the numerical method for the strong solution of SDEs, which is an extension of the formula for weak solution and is equivalent to the theorem given by Higham. We prove that the implicit Euler method is asymptotically stable for some \( p \) and \( q \). To our knowledge, this is the first positive result for the stability property of the implicit Euler method. As the asymptotical stability properties have a strong relationship to the number of the steps, the definition of \( T(A) \)-stability and the \( T(A) \)-stability regions of the Euler methods are also given in this paper.

It should be noticed that the implicit Euler method may be not suitable for nonlinear SDEs or for SDE systems although it is asymptotically stable for some problem parameters of the linear test equation. The reason is that for a given value of \( h \) the probability of the nonlinear equation for \( y_{n+1} \) being a singular nonlinear equation is not very small. The numerical procedure may fail because the iteration
procedure for the nonlinear equations is not convergent. Furthermore, although the implicit Euler method converges as $h \to 0$ to the backward solution, for $b$ large or $b$ very small it can behave like the Itô solution and in this case is a reasonable method.

In order to improve the stability properties of the numerical methods for SDEs, we can consider some modifications to the classic Runge–Kutta methods by introducing a modified Wiener process such as in the work by Mil'shtein, Platen and Schurz [9]. We have also considered the composite Euler method [3] for SDEs. In addition, stepsize control techniques can be used. All of these are the topics of future work.

References
