FINE TOPOLOGY AND GROWTH OF SMALL SUBHARMONIC AND $\delta$–SUBHARMONIC FUNCTIONS IN THE COMPLEX PLANE

A. Yu. Shahverdian, M. Essén and G.S. Hovanessian

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Abstract. The paper gives asymptotical estimates of the behaviour at infinity of small subharmonic or $\delta$–subharmonic functions: this behaviour depends on different kinds of deficiencies. The exceptional sets are described in capacity terms. Our growth restrictions are expressed in integral weight form. We use a generalization of the notion of thin set from potential theory which depends on the corresponding weight function.

1. Introduction

The paper considers the asymptotical behavior of the quantities

$$\frac{w(x)}{T(|x|)} \quad \text{and} \quad \frac{u(x)}{B(|x|)}$$

where $w$ is the difference of two functions subharmonic in the complex plane $\mathbb{C} = \{ |x| < \infty \}$, $u$ is subharmonic in $\mathbb{C}$, $T(r)$ is Nevanlinna's characteristic function of $w$, and $B(r) = \max_{0 \leq \phi \leq 2\pi} u(r\text{e}^{i\phi})$. Essentially, our results are valid for functions of order zero. Estimates of $u(x)/B(|x|)$ were given by J. Littlewood (see the chapter on "A Mathematical Education" in [16]), A. Wiman (see [24] for the early history of these problems), W.K. Hayman [11], G. Valiron [23] and others. The basic problem here is to get a more precise description of exceptional sets, necessarily arising in theorems of this type. In metrical formulation, problems of this type for meromorphic functions have been considered by Y. Kubota [12] (cf. Remark 7 in Section 3). The paper [19] develops further the original approach of Y. Kubota and contains improvements of some results of W.K. Hayman, J. Littlewood, R.P. Boas and others: here the exceptional sets are not characterized in terms of capacity (cf. [11], [14], [16] and [23]).

In the subharmonic case, such characterizations in terms of capacity can be found in the papers [8] by M. Essén, W.K. Hayman and A. Huber, [7] by M. Essén and [18]–[21] by A. Shahverdian. More details on some of these references will be given in Section 3. In the present paper, we introduce methods which can be used to study the $\delta$–subharmonic (or "meromorphic") case: here different kinds of deficiencies appear in our results which is new. Another new aspect is that we use a general comparison function $\omega$ to measure the size of the exceptional set and introduce the notion of $\omega$–thin set that generalizes the concept of thin set in classical thin topology [4]: our function $\omega$ depends on the quantities $T$ and $B$.
which characterize the growth of the functions \( w \) and \( u \). Our exceptional sets are characterized by generalized Wiener conditions involving the function \( \omega \).

2. Some Definitions

We shall work with a modified logarithmic capacity \( \gamma(e) \), defined for subsets \( e \) of \( \mathbb{C}_0 = \mathbb{C} \setminus \{0\} \). Its properties are described in the appendix.

Assuming that the infimum below is positive, we define the capacity \( \gamma(e) \) of a compact set \( e \) contained in \( \mathbb{C}_0 = \mathbb{C} \setminus \{0\} \) in the following way:

\[
\gamma(e) = \left( \inf_{\mu} \int e \int \ln \frac{|x|}{|x - \zeta|} \, d\mu(x) \, d\mu(\zeta) \right)^{-1}.
\]

Here the infimum is taken over all Borel measures \( \mu \) with \( \mu(e) = 1 \) and \( \text{supp}(\mu) \subseteq e \). We define \( \gamma(e) \) first for compact sets and extend it then in the usual way to arbitrary sets. For more details on the definition, we refer to the appendix.

We suppose always that \( w = u - v \) where \( u \) and \( v \) are subharmonic in \( \mathbb{C} \), \( \mu \) is the Riesz measure of \( u \), \( \nu \) is the Riesz measure of \( v \) and \( T \) is the Nevanlinna characteristic function, defined by (cf. e.g. pp. 127 and 508 in [13])

\[
T(r) = \frac{1}{2\pi} \int_0^{2\pi} \max\{0, w(re^{i\phi})\} \, d\phi + N(r, v)
\]

where

\[
N(r, v) = \int_1^r \frac{n\nu(t)}{t} \, dt, \quad n\nu(t) = \nu(|x| \leq t).
\]

We define

\[
\delta(w) = 1 - \limsup_{r \to \infty} \frac{N(r, v)}{T(r)}, \quad \Delta(w) = 1 - \liminf_{r \to \infty} \frac{N(r, u)}{T(r)}.
\]

In the case when \( w = \ln|f| \) and \( f \) is a meromorphic function in \( \mathbb{C} \), the quantities \( \delta(w) \) and \( \Delta(w) \) coincide with the Nevanlinna deficiency \( \delta_f(\infty) \) at \( \infty \) (cf. e.g. p. 709 in [13]) and the Valiron deficiency \( \Delta_f(0) \) at 0, respectively. For \( u \) subharmonic, we introduce

\[
d(u) = 1 - \liminf_{r \to \infty} \frac{N(r, u)}{B(r)}, \quad 0 \leq d(u) \leq 1.
\]

Let \( \omega \) be a positive monotone (increasing or decreasing) function defined in the interval \( (1, \infty) \) and let \( a^+ = \min\{1, a\} \). If \( q > 1 \) is a given number, we consider sets \( e \subseteq \mathbb{C}_0 \) for which the condition

\[
\sum_{n=1}^{\infty} \omega(q^n) \gamma^+(e_n) < \infty
\]

holds. Here \( e_n = e \cap \{q^n \leq |x| \leq q^{n+1}\} \). Using terminology from [4] we call such sets \( \omega \)-thin. It follows from a criterion of Euler that the condition

\[
\sum_{n=1}^{\infty} \omega(q^n) = \infty
\]

holds if and only if

\[
\int_1^{\infty} \frac{\omega(r)}{r} \, dr = \infty.
\]
Hence if (5) is not true, the relation (4) holds for all sets $e$. In particular, it is even possible that $e = \mathbb{C}_0$ (and the statements of Theorems 1 and 2 lose their significance). If (5) holds, then it follows from (4) that $e$ is a small set in an infinite number of rings $q^n \leq |x| \leq q^{n+1}$:

$$\liminf_{n \to \infty} \gamma(e_n) = 0,$$

while for each natural $n$ we have $\gamma^\dagger(\{x : q^n \leq |x| < q^{n+1}\}) = 1$. If $\omega$ increases, then the symbol "$^\dagger$" in (4) can be erased; consequently, for each such $\omega$-thin set $e$ we have

$$\sum_{n=1}^{\infty} \gamma(e_n) < \infty. \quad (6)$$

We note that when $\gamma(e_n)$ is small, $e_n$ must be a small subset of the annulus $\{q^n \leq |x| \leq q^{n+1}\}$ (cf. the appendix for details). The $\gamma$-capacity of the annulus is infinite.

Everywhere below, we suppose that the function $\omega$ satisfies the doubling condition

$$\omega(2r)/\omega(r) = O(1) \quad (r \to \infty) \quad (7)$$

in the case when $\omega$ is increasing. When $\omega$ is decreasing, we assume instead that

$$\omega(r)/\omega(2r) = O(1) \quad (r \to \infty). \quad (7a)$$

3. Main Results

Let us now formulate our main results which include Theorems 1, 2, 4 and Corollaries 9-11. In a certain sense, they generalize or supplement some known results mentioned above. It should be noticed that the major limitations on the growth of ($\delta$-) subharmonic functions are given below in integral weight form. We have not been able to obtain any non-trivial results for functions of positive order. Our theorems are of interest in the study of functions of order zero. Some lower estimates of ($\delta$-) subharmonic functions using capacity, can be found in [18]–[21].

**Theorem 1.** Let $w$ be a $\delta$-subharmonic function and assume that for some monotone function $\omega$ the condition

$$\int_{|x| < \infty} \frac{\omega(|x|)}{T(|x|)} d\mu(x) < \infty \quad (8)$$

holds. Then for an arbitrary number $\theta$ in $(0,1)$, there exists a $\omega$-thin set $e = e_\theta$ such that

$$\liminf_{x \to \infty, T(|x|)} \frac{w(x)}{T(|x|)} \geq \theta \delta(w) - \theta^{-1}\Delta(w). \quad (9)$$

**Theorem 2.** Let $u$ be a subharmonic function and assume that for some monotone function $\omega$ the condition

$$\int_{|x| < \infty} \frac{\omega(|x|)}{B(|x|)} d\mu(x) < \infty \quad (10)$$

holds. Then for an arbitrary number \( \theta \) in \((0,1)\), there exists a \( \omega \)-thin set \( e = e_\theta \) such that

\[
\liminf_{x \to \infty, x \notin e} \frac{u(x)}{B(|x|)} \geq 1 - \theta^{-1}d(u).
\]  
(11)

**Remark 3.** If \( \delta(w) > \Delta(w) \), the right hand member in (9) will be positive for \( \theta \) close to 1. Similarly, the right hand number in (11) will be positive for \( \theta \) close to 1 if \( d(u) < 1 \).

**Theorem 4.** Let \( u \) be a subharmonic function such that

\[
B(r) = N(r) + O(1) \quad (r \to \infty)
\]

and assume that for some monotone \( \omega \) the condition

\[
\int_{|x|<\infty} \omega(|x|)d\mu(x) < \infty
\]  
(12)

holds. Then there exists a \( \omega \)-thin set \( e \) such that

\[
u(x) = B(|x|) + O(1) \quad (x \to \infty, x \notin e).
\]  
(13)

**Remark 5.** It follows from proof of Theorem 4, that if a subharmonic function satisfies the condition \( B = N + o(1) \), then for a given positive number \( \varepsilon \) the exceptional set \( e \) in (13) can be constructed in such a way that the absolute value of the bounded quantity \( O(1) \) in (13) does not exceed \( \varepsilon \) for all \( x \) which are not in \( e \).

It is not difficult to prove the following statement, which gives conditions on \( w \) (or \( u \)) when (8) (or (10)) is true (the proof will be given in Section 4).

**Lemma 6.** Suppose that for a \( \delta \)-subharmonic \( w \) (or subharmonic \( u \)) and some function \( \phi(r) \geq k > 1 \), we have

\[
T(r \phi(r)) = O(T(r)) \quad (or \quad B(r \phi(r)) = O(B(r))).
\]  
(14)

If \( \omega \) is decreasing and

\[
\int_1^\infty \frac{1}{\ln^2 \phi(r)} \frac{\omega(r)}{r} dr < \infty
\]  
(15)

then (8) (or (10)) holds. If \( \omega \) is increasing, (15) holds and

\[
\omega(r) = O(\ln \phi(r)), \quad r \to \infty,
\]  
(16)

then (8) or (10) holds.

If \( \omega \) is increasing and

\[
\int_1^\infty \frac{T(r^2)}{\ln^3 r} \frac{\omega(r)}{r} dr < \infty \quad (or \quad \int_1^\infty \frac{B(r^2)}{\ln^3 r} \frac{\omega(r)}{r} dr < \infty),
\]  
(17)

then (8) (or (10)) holds.

A function \( \phi(r) \) satisfying (14) can be defined in the following way. If \( w \) is a given \( \delta \)-subharmonic function and \( C \) is a given (big) constant, we define

\[
\phi(r) = \sup\{t > 1 : T(rt) \leq CT(r)\}.
\]

If \( T(r) \sim (\ln r)^\alpha \) for some \( \alpha \geq 1 \), we can choose \( \phi(r) \sim r \) (or \( \phi(r) \sim r^\gamma \) for some positive \( \gamma \)). If \( T(r) \) grows in a less regular way, other choices of \( \phi \) may be useful. A similar remark holds for subharmonic functions and \( B(r) \).
Remark 7. To describe the results of Kubota [14] mentioned in the introduction, we consider a measurable subset \( e \) of \((0, \infty)\) and the lower density \( \liminf_{r \to \infty} \frac{|e \cap (0, r)|}{r} \) (where \(| \cdot |\) denotes Lebesgue measure). If \( f \) is a meromorphic function of order 0 and \( \delta(\infty, f) \) and \( \Delta(\infty, f) \) denote the Nevanlinna and Valiron deficiencies of \( f \) at infinity (cf. (3)), then Kubota proves that if \( \delta(\infty, f) \) is positive, then

\[
\delta(\infty, f) \leq \liminf_{r \to \infty, r \notin e} \ln \frac{\mu(r, f)}{T(r, f)} \leq \limsup_{r \to \infty, r \notin e} \frac{B(r, f)}{T(r, f)} \leq \Delta(\infty, f),
\]

where \( e \) is a set of lower density 0, \( T(r, f) \) is the Nevanlinna characteristic and \( B(r, f) \) and \( \mu(r, f) \) denote the maximum and the minimum modulus of \( f \). The exceptional set is here a union of annuli in the plane and considerably larger than the exceptional sets in Theorems 1, 2 and 4 which are defined in terms of capacity and which depend on a parameter \( \theta \).

Remark 8. To discuss the relation between our results and the work of W. Hayman [11], we say that a set \( e \) is a \( C \)-set if it can be covered by discs \( C_{x_n, r_n} \) with centers \( x_n \) and radii \( r_n \) satisfying the condition \( \sum_{n=1}^{\infty} r_n / |x_n| < \infty \). In [11], Hayman proved that if

\[
B(r) = O(\ln^2 r),
\]

then there exists a \( C \)-set \( e \) such that for \( x \to \infty, x \notin e \), we have

\[
u(x) = B(|x|) + o(B(|x|)).
\]

Furthermore, if

\[
B(r) = O(\ln r) \quad \text{and then} \quad \int_C d\mu < \infty,
\]

then there exists a \( C \)-set \( e \) such that for \( x \to \infty, x \notin e \), we have

\[
u(x) = B(|x|) + o(1).
\]

It follows from a result of G. Piranian [17] that in Hayman’s theorem the assumption (18) cannot be replaced by the condition \( B = O(\psi) \) where \( \lim_{r \to \infty} \psi(r) / \ln^2 r = \infty \). It can be proved (see [18], [19]) that a 1-thin set is a \( C \)-set and that there exist \( C \)-sets that are not 1-thin. The set

\[
\bigcap_{0<r<\infty} \left\{ \frac{x}{|x|} : x \in e, |x| \geq r \right\}
\]

has zero Lebesgue measure if \( e \) is a \( C \)-set, and has zero logarithmic capacity if \( e \) is 1-thin (see [18], [19], [21]).

If \( \theta \) is a given number in the interval \((0, 1)\), let us consider the set \( e = \{ x \in C : u(x) \leq \theta B(|x|) \} \). If (19) holds, Arsove and Huber [2] have proved that a Wiener condition (see [4]) holds for \( e \) at \( \infty \) (cf. also Theorem 4 in Essén, Hayman and Huber [8]). In other words, \( e \) is thin at infinity which means that

\[
\sum_{n=1}^{\infty} n \gamma(e_n) < \infty
\]

(cf. (6.2) in Essén and Jackson [9]). Essén gave in [7] further results on this problem in the case \( B = O(\psi) \) where \( \psi \) is a slowly oscillating function which is
such that \( \limsup_{r \to \infty} \psi(r)/\ln^2 r = \infty \). For a result concerning entire functions with slowly oscillating Nevanlinna characteristic, see [21].

As examples of applications of Lemma 6, we consider functions \( w \) or \( u \) of order zero. If (14) holds with \( \phi(r) = r \), we can choose \( \omega(r) = (\ln r)^\gamma \) for some \( \gamma \in [0,1) \) in (15). Applying Theorem 1 or 2 in the case \( \gamma = 0 \), we conclude that there exists a \( 1 \)-thin exceptional set \( e \) such that (9) or (11) holds. We note that formally, formula (4) with \( \omega \) constant coincides with a condition for minimal thinness in a Stolz domain in a half-plane given by H. Jackson (cf. (1.3) in [9]). Note also, that such sets arise in lower estimates of (\( \delta^- \)) subharmonic functions (see [20]). If the behavior of \( w \) or \( u \) is such that we can only say that (14) holds with \( \phi(r) = \ln r \), we can choose \( \omega(r) = (\ln r)^{-1} \) in (15). Thus a more irregular behavior of \( w \) or \( u \) will give us a larger exceptional set.

Conditions (15) and (16) can be applied to functions for which \( T(r) \) or \( B(r) \) grow as \( (\ln r)^\alpha \) for some \( \alpha \geq 1 \). Condition (17) works only if \( 1 \leq \alpha < 2 \).

For functions which grow very slowly, we have the following corollaries of Theorems 1, 2, 4 and Lemma 6 (proofs will be given in Section 4).

In Corollary 11 we consider the special case of a logarithmic potential in the plane \( \mathbb{C} \),

\[
    u(x) = u_\mu(x) = \int_\mathbb{C} \ln |x - \zeta| d\mu(\zeta).
\]

We note that the relation (25) below follows also from results in [2].

**Corollary 9.** If \( u \) is subharmonic and we have the relation \( B(r) = O(\ln^\alpha r)(r \to \infty) \) where \( 1 \leq \alpha < 2 \), then for arbitrary \( \beta \) in \( (0,2-\alpha) \), there exists a set \( e \) in the complex plane \( \mathbb{C} \) such that for \( x \to \infty, x \notin e \),

\[
    \sum_{n=1}^{\infty} n^\beta \gamma(e_n) < \infty \quad \text{and} \quad u(x) = B(|x|) + o(B(|x|)). \tag{21}
\]

**Corollary 10.** If \( u \) is subharmonic and the relation (19) holds, then for each function \( \omega \) satisfying the condition

\[
    \int_\mathbb{C} \omega d\mu < \infty \tag{22}
\]

there exist sets \( e^{(1)} \) and \( e^{(2)} \) such that for \( x \to \infty, x \notin e^{(1)} \)

\[
    \sum_{n=1}^{\infty} n \omega(q^n) \gamma(e_n^{(1)}) < \infty \quad \text{and} \quad u(x) = B(|x|) + o(B(|x|)) \tag{23}
\]

and for \( x \to \infty, x \notin e^{(2)} \)

\[
    \sum_{n=1}^{\infty} \omega(q^n) \gamma(e_n^{(2)}) < \infty \quad \text{and} \quad u(x) = B(|x|) + O(1). \tag{24}
\]

**Corollary 11.** If \( u(= u_\mu) \) is a logarithmic potential and \( u_\mu(0) \neq \infty \) (which implies that \( \int \ln^+ |\zeta| d\mu(\zeta) < \infty \)), then there exist sets \( e^{(1)} \) and \( e^{(2)} \) such that for \( x \to \infty, x \notin e^{(1)} \)

\[
    \sum_{n=1}^{\infty} n^2 \gamma(e_n^{(1)}) < \infty \quad \text{and} \quad u(x) = n_\mu(|x|) \ln |x| + o(n_\mu(|x|) \ln |x|), \tag{25}
\]
and \( e^{(2)} \) is thin at infinity (cf. (20)) and for \( x \to \infty, x \not\in e^{(2)} \)

\[
 u(x) = n_\mu(|x|) \ln |x| + O(1). \tag{26}
\]

From Remark 5 and the proofs of the corollaries, we see that if \( \varepsilon \) is a given positive number, we can choose the exceptional sets \( e^{(2)} \) in Corollaries 10 and 11 in such a way that the absolute values of \( O(1) \)–terms in (24) and (26) do not exceed \( \varepsilon \). In other words, if we consider the collection of sets (compare the notion of fine topology defined in [4]),

\[ \mathcal{T}_\omega = \{ C \setminus e : e \text{ is } \omega\text{-thin} \} \]

then (24) can be rewritten in the more convenient form:

\[
 u(x) = B(|x|) + \varepsilon(x) \quad \text{where} \quad \mathcal{T}_\omega - \lim_{x \to \infty} \varepsilon(x) = 0
\]

and \( \omega \) satisfies (22); such notation can be also used for (26):

\[
 u(x) = n_\mu(|x|) \ln |x| + \varepsilon(x) \quad \text{where} \quad \mathcal{T}_\omega - \lim_{x \to \infty} \varepsilon(x) = 0
\]

and \( \omega = \ln r \). All the statements of Theorems 1, 2, 4 and Corollaries 9–11 can be reformulated in these fine topological terms or, as in [20] and [21], in terms of convergence on filters. For instance, we can write relation (9) from Theorem 1 in the form

\[
 \mathcal{T}_\omega - \lim \inf_{x \to \infty} \frac{w(x)}{T(|x|)} \geq \delta - \Delta.
\]

After applying the transformation \( x \to (x - \zeta)^{-1} (\zeta \neq \infty) \), the condition (20) mentioned in Corollary 11 coincides with the Wiener criterion for irregularity of a point \( \zeta \) in the Dirichlet problem. This statement is in fact an analogy for case of a logarithmic potential (for case of Riesz potentials see [20]) of a theorem of H. Cartan [6] (see also [4], [15], [20]) on the continuity of a Newton potential in the classical fine topology.

4. Proofs of Results

Let us first introduce some notions and definitions which will be used in proofs. Everywhere below

\[
 D_r = \{ |x| \leq r \}, \quad [x, \xi]_r = \frac{r(x - \xi)}{r^2 - x\xi}, \quad D_{x,r} = \{ \xi : [x, \xi] < r \}
\]

\[
 (D_1 = D, \quad [x, \xi]_1 = [x, \xi]).
\]

We will consider the Green potential of a Borel measure \( \mu \geq 0 \) defined in \( D_r \)

\[
 u^\mu(x, r) = \int_{D_r} \ln \frac{1}{[x, \xi]_r} d\mu(\xi), \quad u^\mu(x, 1) = u^\mu(x).
\]

The Green capacity \( C_{g,r} \) of compact subsets \( e \subset D_r \) is defined by

\[
 C_{g,r}(e) = \left( \inf \int \int e \ln \frac{1}{[x, \xi]_r} d\mu(x)d\mu(\xi) \right)^{-1}, \quad (C_{g_1} = C_{g}), \tag{27}
\]

where the infimum is taken over all Borel measures \( \mu \) for which \( \mu(e) = 1 \) and \( \text{supp}(\mu) \subset e \). If \( w^+ = \max(w, 0) \), \( w_+ = (-w)^+ \), \( w = u - v \) and

\[
 m^+(w, r) = \frac{1}{2\pi} \int_{0}^{2\pi} w^+(r e^{i\phi}) d\phi, \quad m_+(w, r) = \frac{1}{2\pi} \int_{0}^{2\pi} w_+(r e^{i\phi}) d\phi
\]
then it follows from Nevanlinna's theorem

\[ T_w(r) = m^+(w, r) + N(r, y) = m_+(w, r) + N(r, u) + w(0) \]

that for the quantities (3) we have

\[ \delta(w) = \lim \inf_{r \to \infty} \frac{m^+(w, r)}{T_w(r)}, \quad \Delta(w) = \lim \sup_{r \to \infty} \frac{m_+(w, r)}{T_w(r)}. \]

In the Proof of Lemma 14, we will need two remarks. We note that the “triangle inequality” for the pseudohyperbolic distance \([x, \xi]\) follows from

\[ \frac{|x, z| - |z, \xi|}{1 - |x, z||z, \xi|} \leq |x, \xi| \leq \frac{|x, z| + |z, \xi|}{1 + |x, z||z, \xi|} \]

which holds if the moduli of \(x, z, \xi\) are all either smaller than 1 or larger than 1 (cf. formula 1.10 in [10]).

**Remark 12.** If \(0 \leq s \leq \lambda < 1\) and \(x\) and \(y\) are in \(D\), it follows from (28) that if \([x, \xi] < s\) and \([\xi, t] < [\lambda, s]\) then \([x, t] < \lambda\). Thus,

\[ \bigcup_{t \in D_{x,s}} D_{t, [\lambda, s]} \subset D_{x, \lambda}. \] (29)

**Remark 13** (see also [21]). If \(\mu\) is a unit measure in \(D\) and \(\lambda\) is an arbitrary number in \((0, \infty)\) then

\[ C_g\left( \{x \in D : u^\tau(x) > \lambda\} \right) \leq \frac{1}{\lambda}. \] (30)

**Proof of Remark 13.** Since \(u^\tau\) is a lower semicontinuous function, \(G = \{x : u^\tau(x) > \lambda\}\) is an open set. Let \(e \subset G\) be a compact set (we can assume \(C_g(e) > 0\)), and let \(\nu\) be a minimizing measure associated with \(e\) ([13]). By Fubini's theorem, we know that

\[ \int_e u^\tau d\nu = \int_D u^\nu d\mu. \]

The left-hand integral is greater than \(\lambda\) because \(e \subset G\). According to Frostman's theorem (cf. p. 60 in [22] or Theorem 5.8 in [13]), we have \(u^\nu(x) \leq (C_g(e))^{-1}\) for every \(x \in D\). Therefore the right-hand integral is less than \((C_g(e))^{-1}\). Remark 13 is proved.

**Lemma 14.** Let \(0 < r < \infty\), let \(\mu\) be a finite Borel measure in \(D_r\) and let for \(\tau > 0\)

\[ e^{(\tau)} = \{x \in D_r : u^\mu(x, r) > \tau\}. \]

Then for each \(\rho > r\) \((\cosh(\tau/3))^{-1}\) and

\[ \rho_\tau = r[\rho/r, 2(\cosh(\tau/3))^{-1}] \]

the inequality

\[ C_g\left( e^{(\tau)} \right) \leq k(\tau) \mu(\{\rho_\tau \leq |x| < r\}) \] (31)

holds, where \(0 < k(\tau) = O(\tau^{-1})(\tau \to +\infty)\).

**Proof of Lemma 14.** It is enough to prove Lemma for the case \(r = 1\) – the general statement can be obtained via the transformation \(x \to rx, x \in D\). It is also clear that we can assume \(\mu(D) = 1\). It is easy to see that for arbitrary \(\nu\) with \(\nu(D) \leq 1, 0 < \eta < 1\), and every \(x \notin \bigcup_{t \in S_\nu} D_{t, \eta}\) (here \(S_\nu = \text{supp}(\nu)\))

\[ u^\nu(x, 1) \leq c(\eta) = \ln \frac{1}{\eta} \] (32)
Indeed, for such $x$ and every $\xi \in S_\nu$, $g(x, \xi) = \ln 1/|x, \xi| \leq c(\eta)$. If $\tau$ is given, we define numbers $0 < \alpha < \beta < 1$ by
\[
\alpha = \left[ \exp \left( -\frac{\tau}{2} \right), \exp \left( -\frac{\tau}{3} \right) \right], \quad \beta = \exp \left( -\frac{\tau}{3} \right).
\] (33)
It is not difficult to check that $c([\alpha, \beta]) \leq \tau/2$. The system of discs $\{D_{x,\alpha}\}_{|x| < 1}$ covers $D$ and satisfies Ahlfors' lemma (cf. Lemma III.3.2 and the remark at the bottom of p. 407 in [15]); then we have a countable collection of hyperbolic discs $\sigma = \{D_{x,\alpha}\}_{n=1}^\infty$ which covers $D$ and which is such that each point in $D$ is contained in at most 6 discs from $\sigma$ (consequently, the set $\{x_n\}_n^\infty$ does not have any cluster points in $D$). Therefore, the system $\sigma_1 = \{D_{x,\alpha}\}_{n=1}^\infty$ has finite multiplicity $k_0$,
\[
k_0 = k_0(\tau) \leq 6(1 - \beta^2)^{-1} \left( 1 + \frac{\beta}{\alpha} \right)^2.
\] (34)
Indeed, if $x \in D$ is contained in $p \geq 1$ discs $D_{x,\xi_{i,\alpha}} \in \sigma$, then for each $i \xi_i \in D_{x,\beta}$. Let $\alpha_0 = (\alpha + \beta)/(1 + \alpha \beta)$. According to (29) with $[\lambda, s] = [\alpha_0, \beta] = \alpha$, we have for every $i$
\[
D_{x,\alpha_i} \subset D_{x,\alpha_0}.
\]
Consequently the sum of the non–Euclidean surface measures of the discs $D_{x,\alpha_i} \subset D_{x,\alpha_0}$ does not exceed $6s$, where
\[
s = 4\pi \frac{\alpha_0^2}{1 - \alpha_0^2}
\]
is the non–Euclidean surface measure of $D_{x,\alpha_0}$. It follows that
\[
p(1 - \alpha^2) \leq 6\alpha_0^2/(1 - \alpha_0^2)
\]
which immediately gives (34). From (33) it is easy to see that
\[
k_0(\tau) = O(1) \quad (\tau \rightarrow +\infty).
\] (35)
Let $\mu_n$ be the restriction of $\mu$ to the disc $D_{x,\alpha_i}$ and let $\nu_n$ be the restriction of $\mu$ to the complement $D_{x,\alpha_i}^C$. We define $e_n^{(\tau)} = e^{(\tau)} \cap D_{x,\alpha_i}$. If $x \in e_n^{(\tau)}$ and $y \notin D_{x,\alpha_i}$, then by (28) $[x, y] \geq [\alpha, \beta]$. Applying (32) with $\eta = [\alpha, \beta]$, it follows that
\[
u^{(\tau)}(x) \leq c([\alpha, \beta]) \leq \tau/2, \quad x \in e_n^{(\tau)}.
\] (36)
Since $\nu^{(\tau)} = \mu - \nu^{(\tau)}$, it follows from (36) and the definition of $\nu^{(\tau)}$ that
\[
u^{(\tau)}(x) \geq \tau/2, \quad x \in e_n^{(\tau)}.
\]
Hence $e_n^{(\tau)} \subset \{x \in D_{x,\alpha} : \nu^{(\tau)}(x) > \tau/4\}$ which by (30) implies that
\[
C_g(e_n^{(\tau)}) \leq \frac{4}{\tau} \mu(D_{x,\alpha_i}).
\]
Let $M_n = \max\{|x| : x \in D_{x,\alpha_i}\}$. If $\rho \in (0, 1)$ is arbitrary, then
\[
C_g \left( e^{(\tau)} \cap (\rho \leq |x| < 1) \right) \leq \sum_{M_n \geq \rho} C_g(e_n^{(\tau)}) \leq \frac{4}{\tau} \sum_{M_n \geq \rho} \mu(D_{x,\alpha_i})
\]
\[
\leq \frac{4k_0}{\tau} \mu \left( \bigcup_{M_n \geq \rho} D_{x,\alpha_i} \right) = k(\tau) \mu \left( \bigcup_{M_n \geq \rho} D_{x,\alpha_i} \right)
\] (37)
where \( k(\tau) = 4k_0\tau^{-1} \). By (35), we have \( k(\tau) = O(\tau^{-1}), \tau \to +\infty \). Let \( D_{\xi,\beta} \) be a hyperbolic disc which touches the euclidean circle \( C_{0,\rho} \) from the inside. Then \( 0 \not\in D_{\xi,\beta} \) if and only if
\[
\rho > \frac{2\beta}{1 + \beta^2} \left( = \left( \frac{1}{3} \right) \sum_{n=1}^{\infty} \frac{1}{n^2} \right).
\]
Furthermore, if this condition holds, the euclidean distance from \( D_{\xi,\beta} \) to the origin is
\[
\rho' = [\rho, \beta] > [\rho, 2\beta] = \rho_r.
\]
It follows that
\[
\bigcup_{\rho \geq \rho_r} D_{\xi_n,\beta} \subset \{ \rho_r < |x| < 1 \}.
\]
Now (37) implies (31) and Lemma 14 is proved.

**Proof of Lemma 6.** For \( R > r \) and \( A(r) = rT'(r) \) (see [12]) we have
\[
T(R) = \int_{r}^{R} \frac{A(t)}{t} \, dt \geq \int_{r}^{R} \frac{A(t)}{t} \, dt \geq A(r) \ln \frac{R}{r}
\]
(38)
\[
T(R) \geq N(R) = \int_{r}^{R} \frac{n(t)}{t} \, dt \geq \int_{r}^{R} \frac{n(t)}{t} \, dt \geq n(r) \ln \frac{R}{r};
\]
choosing \( R = r\phi(r) \geq rk \) and using (14), we obtain
\[
\frac{A(r)}{T(r)} \leq \frac{\text{const}}{\ln \phi(r)} \quad \text{and} \quad \frac{n(r)}{T(r)} \leq \frac{\text{const}}{\ln \phi(r)}.
\]
(39)
Since \( \omega \) decreases, applying (39), after an integration by parts, we obtain that the two integrals
\[
\int_{r}^{\infty} \frac{\omega(r)}{T(r)} \, dn(r) \quad \text{and} \quad \int_{r}^{\infty} \frac{n(r)}{T(r)} \frac{A(r)}{r} \frac{\omega(r)}{T(r)} \, dr
\]
(40)
converge or diverge simultaneously. The proof in the subharmonic case is similar with \( T \) replaced by \( B \) and \( A(r) \) by \( rB'(r) \). This proves that (8) follows from (15). A similar argument holds in the case when \( \omega \) is increasing and (15) and (16) hold.

Now assume that \( \omega \) increases and that (17) holds. For non-constant \( \omega \) we have \( \lim \inf_{r \to \infty} T(r)/\ln r > 0 \) (see [13]). Again integrating by parts we obtain
\[
\text{const.} \int_{1}^{\infty} \frac{\omega(r)}{T} \, dn \leq \int_{1}^{\infty} \frac{\omega}{\ln r} \, dn = \frac{n\omega}{\ln r} \, \left( 1 - \int_{1}^{\infty} \frac{n}{\ln r} \, dw + \int_{1}^{\infty} \frac{n\omega}{r\ln^2 r} \, dr \right).
\]
(41)
Taking \( R = r^2 \) in (38), we see that \( n(r) \leq T(r^2)/\ln r \) – hence the convergence of the last integral in (41) follows from (17). Furthermore, we have
\[
\int_{r}^{\infty} \frac{n\omega}{s \ln^2 s} \, ds \geq n(r)\omega(r)/2 \ln r.
\]
Consequently, \( n(r)\omega(r)/\ln r \to 0 \) as \( r \to \infty \). It is now clear that (8) follows from (17). If \( \omega \) increases, (17) holds and \( u \) is non-constant, we have \( \lim \inf_{r \to \infty} B(r)/\ln r > 0 \). A similar argument shows that (10) follows from (17). This completes the proof of Lemma 6.
Proof of Theorem 1. Let $\mu$ be the Riesz measure of the function $u$, let $\tau_n \geq 1$ be given numbers, and let the number $q$ be such that for every large natural number $n \ (n \geq n_1)$$1 < q < \left(\frac{1}{2} \cosh \left(\frac{\tau_n}{3}\right)\right)^{1/2}$. (42)

Define $r_n = q^{n+2}$ and $e = \bigcup_{n=1}^{\infty} e_n$ where for $n \geq 1$
\[ e_n = \{x \in D_{r_n} : u^\mu(x, r_n) \geq \tau_n\} \cap \{q^n \leq |x| \leq q^{n+1}\}. \] (43)

For $\rho_n = q^n$ and every large $n \ (n \geq n_2)$, we have the inequality $\rho_n \geq r_n \left(\cosh(\tau_n/3)\right)^{-1}$ and hence from (31)
\[ C_{\mu e_n}(e_n) \leq \frac{c_1}{\tau_n} \mu(I_n) \quad (n \geq n_2) \] (44)

where the constant $c_1$ does not depend on $n$ and
\[ I_n = \{\theta_n q^{n+2} \leq |x| \leq q^{n+2}\}, \quad \theta_n = [q^{-2}, 2 \left(\cosh\left(\frac{\tau_n}{3}\right)\right)^{-1}]. \] (45)

Since $e_n \subset \{q^n \leq |x| < q^{n+1}\}$, then for every $\xi, \eta \in e_n$ we have
\[ g(\xi, \eta, r_n) - \ln \frac{|\xi|}{|\xi - \eta|} \leq c_2 \quad (= \ln 2q^2). \]

From (2) and (27) it follows easily that
\[ C_{\mu e_n}(e_n) \geq \frac{\gamma(e_n)}{1+c_2 \gamma(e_n)} \geq c_3 \gamma^+(e_n) \]
with some positive constant $c_3 \ (= (1+c_2)^{-1})$. From (44), we have
\[ \gamma^+(e_n) \leq \frac{c_4}{\tau_n} \mu(I_n) \quad (n \geq n_2). \] (46)

where again the constant $c_4 > 0$ does not depend on $n$.

Let $\theta \in (0, 1)$ be the number in the statement of the theorem and let
\[ q = \frac{1+\theta}{1-\theta}. \] (47)

Below we let $\tau_n \to \infty$. Thus for $n_1$ sufficiently large we know that (42) holds for $n \geq n_1$. Let $e_\theta$ be the set $e$ defined above using (43). We define also $I_n^1 = \{q^{n-1} \leq |x| \leq q^{n+2}\}$.

Assume that (8) holds for a given function $w$. Then there exists a function $\sigma_1(r) = o(1) \ (r \to \infty)$ and numbers $\sigma_n = \max\{\sigma_1(r) : r \in I^1_n\}$ such that
\[ \int_{|x|<\infty} \frac{1}{\sigma_1(|x|)} \frac{\omega(|x|)}{T(|x|)} \ d\nu(x) < \infty, \quad \lim_{n \to \infty} \sigma_n T(q^{n+2}) = +\infty. \] (48)

In (42) and (45), we choose $\tau_n = \sigma_n T(q^{n+2})$. It is easy to check that $\lim_{n \to \infty} \theta_n = q^{-2}$. Thus, for $n \geq n_3$ we have $q^{-3} \leq \theta_n < 1$. Hence from (46)
\[ \gamma^+(e_n) \leq \frac{2c_4}{\tau_n} \mu(I_n^1). \] (49)
We have proved that
\[ \omega(q^{n+2})\gamma^+(e^n) \leq c_4 \frac{\omega(q^{n+2})\mu(I_n^1)}{\sigma_n T(q^{n+2})} \leq c_4 \int_{I_n^1} \frac{1}{\sigma(|x|)} \frac{\omega(|x|)}{T(|x|)} d\mu(x). \] (50)

Since the system of rings \( \{I_n^1\}_{n=1}^\infty \) has finite multiplicity, it follows from (7) and (8) that the series in (4) is convergent.

Let us now prove inequality (9). If \( x \in \mathbb{C} \) is an arbitrary point, then we have according to the Poisson–Jensen formula in \( D_r \) that
\[ w(x) = I(x) - u^+(x, r) + u^-(x, r) \geq I(x) - u^+(x, r), \] (51)
where
\[ I(x) = \frac{1}{2\pi} \int_0^{2\pi} w(re^{i\phi})Pd\phi = \frac{1}{2\pi} \int_0^{2\pi} w^+ Pd\phi - \frac{1}{2\pi} \int_0^{2\pi} w^- Pd\phi \] (52)
and \( P = P(x, r, \phi) \) is the Poisson kernel:
\[ \frac{r - |x|}{r + |x|} \leq P(x, r, \phi) = \frac{r^2 - |x|^2}{|re^{i\phi} - x|^2} \leq \frac{r + |x|}{r - |x|}. \]

It follows from (51) and (52) that for all \( x \in D_r \),
\[ w(x) \geq m_+(r, w) - m_- (r, w) - \frac{r + |x|}{r - |x|} m_+(r, w) - u^+(x, r). \] (53)

If now \( x \notin e \) is an arbitrary point, there exists \( n = n(x) \) (\( n(x) \to +\infty \)) such that \( q^n \leq |x| < q^{n+1} \) and \( x \notin e_n \); from (53) with \( r = r_n = q^{n+2} \) we have
\[ \frac{r_n - |x|}{r_n + |x|} \geq \theta, \quad \frac{r_n + |x|}{r_n - |x|} \leq \theta^{-1}, \quad u^+(x, r_n) \leq \tau_n = \sigma_n T(r_n). \] (54)

It follows from (53) that
\[ w(x) \geq \theta m_+(r_n, w) - \theta^{-1} m_+(r_n, w) - \sigma_n T(r_n). \]

Dividing by \( T(r_n) \) and letting \( x \to \infty \) outside \( e \), we obtain (9). This concludes the Proof of Theorem 1.

**Proof of Theorem 2.** Assume that (10) holds for a given function \( u \). Then there exists a function \( \sigma_2(r) = o(1) \) (\( r \to \infty \)) and numbers \( \sigma_n = \max\{\sigma_2(r) : r \in I_n^1\} \) such that
\[ \int_{|x|<\infty} \frac{1}{\sigma_2(|x|) B(|x|)} d\mu(x) < \infty, \quad \lim_{n \to \infty} \sigma_n B(q^{n+2}) = +\infty. \]

In (42), (43) and (45), we choose \( \tau_n = \sigma_n B(q^{n+2}) \) in the same way as in the Proof of Theorem 1 to obtain the convergence of the series in (4). Let us now prove inequality (11). Using the inequality \( N(r) \leq B(r) \) and (51) we have that for all \( x \in D_r \),
\[ I(x) = B(r) - \frac{1}{2\pi} \int_0^{2\pi} (B(r) - u(re^{i\phi})) Pd\phi \geq \] \[ B(r) - \frac{r + |x|}{r - |x|} \frac{1}{2\pi} \int_0^{2\pi} (B(r) - u(re^{i\phi})) d\phi = B(r) - \frac{r + |x|}{r - |x|} (B(r) - N(r)). \] (55)
Again, it follows from (51) and (55) that for all \( x \not\in D_r \)

\[
u(x) \geq B(r) \left[ 1 - \frac{r + |x|}{r - |x|} \left( 1 - \frac{N(r)}{B(r)} \right) \right] - u^\mu(x, r). \tag{56}
\]

If \( x \not\in e \) then there exists \( n = n(x) \) such that \( q^n \leq |x| < q^{n+1} \) and \( x \not\in e_n \); from (56) with \( r = r_n = q^{n+2} \), we obtain

\[
\frac{r_n + |x|}{r_n - |x|} \leq \theta^{-1}, \quad u^\mu(x, r_n) \leq \tau_n = \sigma_n B(r_n). \tag{57}
\]

It follows from (56) that

\[
u(x) \geq B(r_n) \left[ 1 - \theta^{-1} \left( 1 - \frac{N(r_n)}{B(r_n)} \right) \right] - \sigma_n B(r_n).
\]

Dividing by \( B(r_n) \) and letting \( x \to \infty \) outside \( e \), we obtain (11). This concludes the Proof of Theorem 2.

**Proof of Theorem 4.** Assume that (12) holds for a given function \( u \). In (42) and (45), we choose \( \tau_n = c \) where \( c \) is such that \( \theta_n = [q^{-2}, 2\left( \cosh(c/3) \right)^{-1}] > q^{-3} \). From (46), we see that

\[
\omega(q^{n+2})\gamma^1(e_n) \leq c_5 \omega(q^{n+2})\mu(I_n^1) \leq c_5 \int_{I_n^1} \omega d\mu.
\]

Now (12) gives us the convergence of the series from (4).

To prove equation (13), we assume that \( x \not\in e \). Arguing as in the proof of Theorem 2, we define numbers \( n = n(x) \). From (56) with \( r = r_n = q^{n+2} \), we obtain that for \( q^n \leq |x| < q^{n+1} \),

\[
u(x) \geq B(r_n) - \frac{q + 1}{q - 1} (B(r_n) - N(r_n)) - u^\mu(x, r_n).
\]

Using the assumption \( B = N + O(1) \) from the formulation of theorem and the fact that \( u^\mu(x, r_n) \leq \tau_n = c \) for \( x \not\in e \), we see that

\[
u(x) \geq B(r_n) + O(1) \geq B(|x|) + O(1) \quad x \to \infty, \quad x \not\in e,
\]

and (13) is proved. Theorem 4 is proved.

In the proofs of the corollaries, we need some facts about functions of small order (cf. Section 3.5 in [3]). If \( u \) has order zero, then

\[
B(r) \leq N(r) + Q(r),
\]

where

\[
Q(r) = r \int_r^\infty t^{-2}n(t)dt.
\]

(i) If \( 1 \leq \alpha \leq 2 \) and

\[
B(r) = O(\ln^\alpha r), \quad r \to \infty, \tag{58}
\]

then \( Q(r) = O((\ln r)^{\alpha - 1}) \), and \( N(r)/B(r) \to 1 \) as \( r \to \infty \) (note that \( N(r) \leq B(r) \) and that \( \liminf_{r \to \infty} B(r)/\ln r > 0 \)).

(ii) If \( n(\infty) \) is finite, then \( Q(r) \) tends to zero as \( r \to \infty \) and hence

\[
B(r) = N(r) + o(1), \quad r \to \infty. \tag{59}
\]
Proof of Corollary 9. If (58) holds, it is easy to see that (17) holds for \( \omega(r) = \ln^\beta r \) \( (0 \leq \beta < 2 - \alpha) \). It follows from Lemma 6 that inequality (10) holds as well. From the remarks above, it is clear that \( d(u) = 0 \) and (21) follows from Theorem 2. We have proved Corollary 9.

Proof of Corollary 10. From (19), we see that there exist increasing functions \( \omega_0 \) satisfying (22). Furthermore, it is clear from (19) and (22) that (10) holds with \( \omega(r) = \omega_0(r) \ln r \). Again, we have \( d(u) = 0 \) (cf. (59)) and we can deduce (23) from Theorem 2. It is clear that (24) follows from Theorem 4. This concludes the Proof of Corollary 10.

Proof of Corollary 11. Since

\[
\mu_\alpha(0) = \int_0^\infty \ln r \, dn(r)
\]

is finite, we see that (22) holds with \( \omega(r) = \ln r \) (in particular we have \( \mu(\mathbb{C}) < \infty \)) and (24) follows from Corollary 10. From (59) we have \( B(r) = n(r) \ln r + O(1) \) and (26) follows from Theorem 4. Corollary 11 is proved.

Appendix

A Modified Logarithmic Capacity.

The following modified logarithmic capacity \( \gamma(e) \), defined for subsets \( e \subset \mathbb{C}_0 = \mathbb{C} \setminus \{0\} \), was introduced in [18–21]. We consider compact sets \( e \) and the energy integral

\[
I(\mu, e) = \int_e \int_e \ln \frac{|\xi|}{|z - \xi|} \, d\mu(z) \, d\mu(\xi)
\]

(60)

where \( \mu \) is a positive Borel measure with total mass 1 and with supp \( \mu \subset e \) (this convention holds throughout this section). Let

\[
V_\gamma(e) = \inf_{\mu} I(\mu, e).
\]

We define

\[
\gamma(e) = \begin{cases} 
(V_\gamma(e))^{-1} & \text{if } V_\gamma(e) > 0 \\
\infty & \text{if } V_\gamma(e) \leq 0.
\end{cases}
\]

(61)

We shall also need the classical energy integral

\[
J(\mu, e) = \int_e \int_e \ln \frac{1}{|z - \xi|} \, d\mu(z) \, d\mu(\xi)
\]

and let

\[
V_J(e) = \inf_{\mu} J(\mu, e).
\]

The logarithmic capacity \( \text{cap}(e) \) is defined by

\[
\text{cap}(e) = \exp\{-V_J(e)\}
\]

(cf. [1]): it is related to the classical logarithmic potential

\[
u_\mu(z) = \int_e \ln \frac{1}{|z - \xi|} \, d\mu(\xi).
\]
SMALL SUBHARMONIC AND $\delta$-SUBHARMONIC FUNCTIONS

Since $0 \notin e$, it is easy to see that there exists a measure $\mu_I$ such that $V_I(e) = I(\mu_I, e)$ (cf. [5]). We claim that

$$u(\mu_I, z) = \frac{1}{2} \int e \ln \frac{|\xi||z|}{|z - \xi|^2} d\mu_I(\xi) = V_I(e) \quad \text{q.e. on } e,$$  \hspace{1cm} (62)

where q.e. means quasi everywhere, i.e. outside a set of outer logarithmic capacity zero.

In the discussion of our new capacity $\gamma(e)$, the function $u(\mu, z)$ and the symmetrized form of the energy integral $I(\mu, e)$ play the same role as $u_\mu(z)$ and $J(\mu, e)$ did in the discussion of $\text{cap}(e)$ (cf. [5]).

We use the same variational argument as in the proof of Theorem III.3 in [5], applied to our energy integral which can be written in the symmetric form

$$I(\mu, e) = \frac{1}{2} \int e \int e \ln \frac{|z|}{|z - \xi|^2} d\mu(z)d\mu(\xi).$$

1. If $u(\mu_I, z) < V_I(e) - \varepsilon$ on a set $T \subset e$ where $\varepsilon$ is positive and $\text{cap}(T) > 0$, we let $\tau$ be a distribution of unit mass on $T$ such that $u(\tau, z) \leq K$ on $e$ and let $\mu_\delta = (1 - \delta)\mu_I + \delta\tau$ for $0 < \delta < 1$. Then $\mu_\delta(e) = 1$, $\text{supp} \mu_\delta \subset e$ and

$$I(\mu_\delta, e) = (1 - \delta)^2 I(\mu_I, e) + 2\delta \int e u(\mu_I, z)d\tau(z) + O(\delta^2) \leq (1 - 2\delta)V_I(e) + 2\delta(V_I(e) - \varepsilon) + O(\delta^2) < V_I(e),$$

if $\delta > 0$ is small enough. This is impossible and we conclude that $u(\mu_I, z) \geq V_I(e)$ everywhere on $e$ except on a set of zero capacity.

2. Conversely, if $u(\mu_I, z_0) > V_I(e)$ for some $z_0 \in \text{supp} \mu_I$, then $u(\mu_I, z) > V_I(e)$ on a neighborhood of $z_0$ of positive $\mu_I$-measure which contradicts the fact that

$$I(\mu_I, e) = \int e u(\mu_I, z)d\mu_I(z) = V_I(e).$$

We have proved that (62) holds.

**Theorem 15.** Let $d = \text{dist.}(0, e) > 0$. Then

$$V_I(e) = V_J(e) + \ln d + \frac{1}{2} \int e \ln(|\xi|/d)d\mu_I(\xi).$$  \hspace{1cm} (63)

**Corollary 16.** $V_I(e) \geq \ln (d/\text{cap}(e))$.

**Proof of Corollary 16.** This is clear since $V_J(e) = -\ln \text{cap}(e)$ and the integral in Theorem 15 is nonnegative.

**Proof of Theorem 15.** Let $\Omega$ be the unbounded component of $\mathbb{C} \setminus e$. Let $g$ be Green's function in $\Omega$ with a pole at infinity: $g$ is harmonic in $\Omega$, vanishes on $\partial \Omega$ and its asymptotic behaviour at infinity is of the form

$$g(z) = \ln |z| + \rho(\Omega) + \varepsilon(z),$$

where $\rho(\Omega)$ is Robin's constant and $\varepsilon(z) \to 0$ as $z \to \infty$. Let $H$ be a harmonic function in $\Omega$ vanishing at infinity with boundary values $\ln(|z|/d)$ on $\partial \Omega$. As in the proof of Theorem 2.2 in [1], we consider the function

$$G(z) = \int e \ln \frac{|\xi|}{|z - \xi|^2} d\mu_I(\xi) - 2V_I(e) + H(z) + \ln d.$$
It follows from (62) that \( G(z) \) vanishes q.e. on \( e \). Furthermore, as \( z \to \infty \),
\[
-\frac{1}{2} G(z) = \ln |z| + \frac{1}{2} \left(2V_I(e) - \ln d - \int_e \ln |\xi| d\mu_I(\xi)\right),
\]
and it follows that we have \( g(z) = -G(z)/2 \) and thus that
\[
\rho(\Omega) = V_I(e) - \frac{1}{2} \int_e \ln(|\xi|/d) d\mu_I(\xi).
\]
From [1], we know that \( \rho(\Omega) = V_J(e) \). We have proved Theorem 15. \( \square \)

**Proposition 17.** Let \( \max_{z \in e} |z| = D \). If \( \text{cap}(e) < d \), then
\[
\left(\ln \left(\frac{D}{\text{cap}(e)}\right)\right)^{-1} \leq \gamma(e) \leq \left(\ln \left(\frac{d}{\text{cap}(e)}\right)\right)^{-1}.
\]

**Proof.** From Theorem 15, we see that
\[
\ln \left(\frac{d}{\text{cap}(e)}\right) \leq V_I(e) \leq \ln \left(\frac{d}{\text{cap}(e)}\right) + \frac{1}{2} \ln(D/d) \leq \ln \left(\frac{D}{\text{cap}(e)}\right),
\]
which proves Proposition 17. \( \square \)

**Remark 18.** This kind of estimate is of interest when we know apriori that our set \( e \) is contained in an annulus \( A_n = \{z \in \mathbb{C} : q^n \leq |z| \leq q^{n+1}\} \) for a given number \( q > 1 \) and an integer \( n \). If \( \gamma(e) \) is small, it follows from Proposition 17 that \( \text{cap}(e) \) must be small. Since \( D/d \leq q \), we deduce from (64) that
\[
\left(\ln q + \ln \left(\frac{d}{\text{cap}(e)}\right)\right)^{-1} \leq \gamma(e) \leq \left(\ln \left(\frac{d}{\text{cap}(e)}\right)\right)^{-1}.
\]

In the case when \( 0 \in \Omega \), we can also estimate \( \gamma(e) \) in terms of \( g(0) \). From [1], we know that
\[
g(0) = \ln \left(\frac{1}{\text{cap}(e)}\right) + \int_e \ln |\xi| d\mu_J(\xi),
\]
and it follows that
\[
\ln \left(\frac{d}{\text{cap}(e)}\right) \leq g(0) \leq \ln(D/\text{cap}(e)).
\]

In the case of an annulus discussed above with \( \gamma(e) \) small, we deduce that
\[
\left(g(0) + \ln q\right)^{-1} \leq \gamma(e) \leq \left(g(0) - \ln q\right)^{-1}.
\]

We note that if \( V_I(e) \) is large, \( \ln \left(\frac{d}{\text{cap}(e)}\right) \) and thus also \( g(0) \) must be large. If \( \gamma(e) \) is small with \( e \) contained in an annulus as above, we have \( \gamma(e) \approx g(0)^{-1} \).

We can also use Proposition 17 to estimate \( \gamma(e) \) for configurations for which \( \text{cap}(e) \) is known. Let us look at some examples of capacities for sets mentioned in Ch. 3.9 in [22]. Since \( \gamma \) is invariant under transformations \( z \to az \), \( (a > 0) \) in the complex plane, it is often sufficient to check a normalized case.

(i) If \( L \) is an arc of a circle centered at the origin of angular measure \( \varphi \in (0, 2\pi) \), then \( \text{cap}(L) = r \sin (\varphi/4) \) and
\[
\gamma(L) = \left(\ln \left(\frac{r}{\text{cap}(L)^{-1}}\right)\right) = \left(\ln \left(\frac{1}{\sin (\varphi/4)}\right)\right)^{-1}.
\]
(ii) All circles \( C = \{ z \in \mathbb{C} : |z - \xi| < r \} \) touching both sides of a given angle in \( \mathbb{C} \) with vertex at the origin and opening \( \varphi \in (0, \pi) \) have the same \( \gamma \)-capacity. Without loss of generality, we choose \( |\xi| = 1 \). Then
\[
d = 1 - \sin (\varphi/2), \quad D = 1 + \sin (\varphi/2), \quad \text{cap}(C) = r = \sin (\varphi/2),
\]
and we obtain the estimate
\[
\left( \ln \left( \frac{1 + \sin (\varphi/2)}{\sin (\varphi/2)} \right) \right)^{-1} \leq \gamma(C) \leq \left( \ln \left( \frac{1 - \sin (\varphi/2)}{\sin (\varphi/2)} \right) \right)^{-1}.
\]

(iii) Let \( S \) be a rectilinear segment \([a, b]\) on the real line with \( 0 < a < b \). Then \( \text{cap}(S) = (b - a)/4 \) and
\[
\left( \ln \left( \frac{4b}{b - a} \right) \right)^{-1} \leq \gamma(S) \leq \left( \ln \left( \frac{4a}{b - a} \right) \right)^{-1}.
\]

For large sets \( e \), \( V_I(e) \) can be negative. As an example, we consider \( e = \{ z \in \mathbb{C} : 1 \leq |z| \leq R \} \) and claim that \( V_I(e) = -\ln R/4 \).

Since \( V_I(e) \) is finite, it follows from Theorem 15 that \( V_I(e) \) is finite and there exists a minimizing measure. Due to the radial symmetry of the set \( e \), it suffices to consider measures of the form
\[
d\mu(z) = d\nu(r) \times (d\theta/2\pi), \quad \mu(e) = 1.
\]
A computation shows that
\[
I(\mu, e) = \int_1^R (\nu(r)^2 - \nu(r)) dr/r.
\]
It is easy to see that \( V_I(e) \) is assumed if \( \nu(r) \) is defined by
\[
\nu(1) = 0, \quad \nu(r) = 1/2, \quad 1 < r < R, \quad \nu(R) = 1,
\]
and we see that \( V_I(e) = -\ln R/4 \). The support of the minimizing measure is contained in the circles \( \{|z| = 1\} \) and \( \{|z| = R\} \).

References


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A. Yu. Shahverdian
Yerevan Physics Institute
Alikhanian Brothers Str. 2 375036
Yerevan
ARMENIA
svrdn@jerewan1.yerphi.am

M. Essén
Department of Mathematics
Uppsala University
PO Box 480
S–751 06 Uppsala
Sweden
EUROPE
matts@math.uu.se

G.S. Hovanessian
Gyumri Branch of Yerevan Engineering University
M. Mkrtchyan Str. 2 377503
Gyumri
ARMENIA
seuagec@shirak.am