SUBCLASSES OF HARMONIC UNIVALENT FUNCTIONS

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Abstract. Complex-valued harmonic functions that are univalent and sense preserving in the unit disk $\Delta$ can be written in the form $f = h + \overline{g}$, where $h$ and $g$ are analytic in $\Delta$. We study subclasses for which $h$ and $g$ have the form $h(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n > 0$ and $g(z) = -\sum_{n=1}^{\infty} b_n z^n$, $b_n > 0$, $b_1 < 1$ and $f$ maps $\Delta$ onto a domain that is either starlike with respect to the origin or convex. Characterizing conditions involving bounds on the coefficients lead to various extremal properties. We conclude with some extremal properties for subclasses having $b_1$ fixed.

1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a domain $D \subseteq \mathbb{C}$ if both $u$ and $v$ are real harmonic in $D$. In any simply connected domain, we can write

$$f = h + g,$$

where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the co-analytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $D$ is that $|h'(z)| > |g'(z)|$ in $D$. See [2].

Denote by $S_H$ the class of functions $f$ of the form (1) that are harmonic univalent and sense-preserving in the unit disk $\Delta = \{z : |z| < 1\}$ with $f(0) = f_{\bar{z}}(0) - 1 = 0$. Thus we may write

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \quad (2)$$

Note that $S_H$ reduces to $S$, the class of normalized univalent analytic functions whenever the co-analytic part of $f$ is zero. For $f = h + \overline{g} \in S_H$, $h'(0) = 1 > |g'(0)| = |b_1|$ implies that the function $(f - b_1 \overline{f})(1 - |b_1|^2)^{-1}$ is also in $S_H$. Consequently, we may sometimes restrict ourselves to $S^0_H$, the subclass of $S_H$ for which $b_1 = f_{\bar{z}}(0) = 0$. Both $S_H$ and $S^0_H$ are normal families; however, only $S^0_H$ is compact. See [2].

Let $S^*_H$ and $K_H$ be the subclasses of $S_H$ consisting of functions $f$ that map $\Delta$ onto starlike and convex domains, respectively. We further denote by $T^*_H$ and $TK_H$ the subclasses of $S^*_H$ and $K_H$, respectively, for which the coefficients of $f = h + \overline{g}$

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take the form

\[ h(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0 \quad \text{and} \quad g(z) = -\sum_{n=1}^{\infty} b_n z^n, \quad b_n \geq 0, \quad b_1 < 1. \quad (3) \]

In [5], the families \( T^0_H = T^*_H \cap S^0_H \) and \( TK^0_H = TK_H \cap S^0_H \) were investigated. Proofs for the compact subfamilies \( S^*_H, S^0_H \), and \( K^0_H \) frequently do not carry over to the corresponding families \( S_H, S^*_H \), and \( K_H \). The harmonic Koebe function, defined in [2], is conjectured to have extremal coefficient bounds for \( S^0_H \). In [3], these bounds are shown to be extremal for \( S^*_H \). See also Section 4 of [1] for extremal properties for \( S^*_H \) and an operator that takes functions from \( S^*_H \) to \( K^0_H \).

In this note, we generalize results for \( T^0_H \) and \( TK^0_H \) to the families \( T^*_H \) and \( TK^*_H \). There are some fundamental differences in the results, because \( T^*_H \), unlike \( T^0_H \), is not a compact family. To see this, note that \( f_n(z) = z - \frac{n}{n+1} \bar{z} \in T^*_H \), for \( n = 1, 2, 3, \ldots \), while \( \lim_{n \to \infty} f_n(z) = z - \bar{z} \) is not even univalent in \( \Delta \).

We will also obtain surprising results for \( T^*_H \) when \( b_1 \) is fixed. In the sequel, we shall take \( f = h + \bar{g} \), where \( h \) and \( g \) are in the form of either (2) or (3).

2. Main Results

We begin with some sufficient coefficient conditions.

**Theorem 2.1.** For \( f = h + \bar{g} \) with \( h \) and \( g \) of the form (2), if

\[ \sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \leq 1, \]

then \( f \) is univalent in \( \Delta \) and \( f \in S^*_H \).

**Proof.** Since \( \sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \leq 1 \), for any \( z \) with \( |z| = r \in (0,1) \), we have that

\[ |h'(z)| \geq 1 - \sum_{n=2}^{\infty} n|a_n|r^{n-1} > \sum_{n=1}^{\infty} n|b_n|r^{n-1} \geq |g'(z)|. \]

Thus, \( f \) is locally univalent and sense preserving. If \( g \equiv 0 \), the univalence of \( f \) will follow from its starlikeness; otherwise, for \( z_1 \neq z_2 \), it follows that
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\[
\frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} = \frac{\sum_{n=1}^{\infty} b_n(z_1^n - z_2^n)}{(z_1 - z_2) + \sum_{n=2}^{\infty} a_n(z_1^n - z_2^n)}
\]

\[
= \frac{\sum_{n=1}^{\infty} b_n(z_1^{n-1} + z_1^{n-2}z_2 + \ldots + z_2^{n-1})}{1 + \sum_{n=2}^{\infty} a_n(z_1^{n-1} + z_1^{n-2}z_2 + \ldots + z_2^{n-1})}
\]

\[
\leq \frac{\sum_{n=1}^{\infty} n|b_n|}{1 - \sum_{n=2}^{\infty} n|a_n|} < 1.
\]

Hence,

\[
\frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} = 1 - \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} > 0
\]

and we conclude that \( f \) is univalent. Finally, starlikeness will follow from showing that \( \frac{\partial}{\partial \theta} (\arg f(re^{i\theta})) > 0 \) for \( 0 \leq \theta < 2\pi, 0 < r < 1 \). Since

\[
\frac{\partial}{\partial \theta} (\arg f(re^{i\theta})) = \Re\left\{ \frac{1 - \bar{b}_1 e^{-i2\theta} + \sum_{n=2}^{\infty} (na_ne^{i(n-1)\theta} - nb_ne^{-i(n+1)\theta})r^{n-1}}{1 + \bar{b}_1 e^{-i2\theta} + \sum_{n=2}^{\infty} (a_ne^{i(n-1)\theta} + \bar{b}_ne^{-i(n+1)\theta})r^{n-1}} \right\}
\]

\[
:= \Re\zeta,
\]

\( \zeta \) will have positive real part if the modulus of

\[
1 - \zeta = 2\bar{b}_1 e^{-i2\theta} - \sum_{n=2}^{\infty} ((n - 1)a_ne^{i(n-1)\theta} - (n + 1)b_ne^{-i(n+1)\theta})r^{n-1}
\]

\[
2 + \sum_{n=2}^{\infty} ((n + 1)a_ne^{i(n-1)\theta} - (n - 1)b_ne^{-i(n+1)\theta})r^{n-1}
\]

is bounded above by 1. From \( \sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \leq 1 \), we see that \( \sum_{n=2}^{\infty} |a_n| \leq \frac{1}{2} \) and \( |b_1| \leq 1 - \sum_{n=2}^{\infty} n(|a_n| + |b_n|) \). Hence,

\[
\sum_{n=2}^{\infty} ((n + 1)|a_n| + (n - 1)|b_n|) = \sum_{n=2}^{\infty} n(|a_n| + |b_n|) + \sum_{n=2}^{\infty} |a_n| - \sum_{n=2}^{\infty} |b_n|
\]

\[
\leq \frac{3}{2} - |b_1| < 2.
\]
It follows that
\[
\frac{|1 - \zeta|}{|1 + \zeta|} < \frac{2|b_1| + \sum_{n=2}^{\infty} ((n - 1)|a_n| + (n + 1)|b_n|)}{2 - \sum_{n=2}^{\infty} ((n + 1)|a_n| + (n - 1)|b_n|)}
\]
\[
< \frac{2 \left( 1 - \sum_{n=2}^{\infty} n(|a_n| + |b_n|) \right) + \sum_{n=2}^{\infty} ((n - 1)|a_n| + (n + 1)|b_n|)}{2 - \sum_{n=2}^{\infty} ((n + 1)|a_n| + (n - 1)|b_n|)}
\]
\[
= 1.
\]
Therefore, \( Rf \zeta > 0 \) as needed.

Corollary 2.2. For \( f = h + g \) with \( h \) and \( g \) of the form (2), if
\[
\sum_{n=2}^{\infty} n^2|a_n| + \sum_{n=1}^{\infty} n^2|b_n| \leq 1,
\]
then \( f \in \mathcal{K}_H \).

Proof. A necessary and sufficient condition for \( f \) to map \( |z| = r \) onto a convex domain is that
\[
\frac{\partial}{\partial \theta} \left( \text{arg} \left\{ \frac{\partial}{\partial \theta} f(re^{i\theta}) \right\} \right)
\]
\[
= \text{Im} \left\{ \frac{\partial}{\partial \theta} \log \left( \frac{\partial}{\partial \theta} f(re^{i\theta}) \right) \right\}
\]
\[
= \Re \left\{ \frac{1 + \overline{b_1}e^{-i2\theta} + \sum_{n=2}^{\infty} (n^2a_ne^{i(n-1)\theta} + n^2b_ne^{-i(n+1)\theta})r^{n-1}}{1 - \overline{b_1}e^{-i2\theta} + \sum_{n=2}^{\infty} (n(a_ne^{i(n-1)\theta} - \overline{b_n}e^{-i(n+1)\theta})r^{n-1}} \right\}
\]
is positive. The remainder of the proof proceeds along the same lines as that of Theorem 2.1.

We next look at the subclasses for which these sufficient conditions are also necessary.

Theorem 2.3. Let \( f = h + g \) where \( h \) and \( g \) are of the form (3). Then \( f \in T^*_H \) if
and only if \( \sum_{n=2}^{\infty} na_n + \sum_{n=1}^{\infty} nb_n \leq 1 \).

Proof. In view of Theorem 2.1, we only need to show that \( f \notin T^*_H \) if the coefficient condition is not satisfied. For this case, we will show that \( f \) is not even univalent. Setting \( z = r > 0 \), we have \( f(r) = (1 - b_1)r - \sum_{n=2}^{\infty} (a_n + b_n)r^n \) and \( f'(r) = (1 - b_1) - \sum_{n=2}^{\infty} n(a_n + b_n)r^{n-1} \). Since \( f'(0) = 1 - b_1 > 0 \) and \( f'(1) < 0 \), there must exist an \( r_0 \), \( r_0 < 1 \), for which \( f'(r_0) = 0 \). Hence, \( r_0 \) is a local maximum for \( f(r) \) and \( f(r) \) is not even one-to-one on the real interval \( (0, 1) \).
Corollary 2.4. For \( f = h + \bar{g} \) where \( h \) and \( g \) satisfy (3), \( f \in \mathcal{T}K_H \) if and only if 
\[ \sum_{n=2}^{\infty} n^2a_n + \sum_{n=1}^{\infty} n^2b_n \leq 1. \]

Proof. In view of the corollary to Theorem 2.1, it suffices to show that \( f \notin \mathcal{T}K_H \) if the coefficient inequality does not hold. For \( f = h + \bar{g} \) where \( h \) and \( g \) satisfy (3), a necessary and sufficient condition for \( f \) to map \( |z| = r \) onto a convex domain is that

\[ \frac{\partial}{\partial \theta} \left( \text{arg} \left\{ \frac{\partial}{\partial \theta} f(re^{i\theta}) \right\} \right) = \Re \left\{ \frac{1 - \sum_{n=2}^{\infty} n^2a_n e^{i(n-1)\theta} r^{n-1} - \sum_{n=1}^{\infty} n^2b_n e^{-i(n+1)\theta} r^{n-1}}{1 - \sum_{n=2}^{\infty} na_n e^{i(n-1)\theta} r^{n-1} + \sum_{n=1}^{\infty} nb_n e^{-i(n+1)\theta} r^{n-1}} \right\} \]

is positive. When \( \theta = 0 \), the last expression is

\[ 1 - \sum_{n=2}^{\infty} n^2a_n r^{n-1} - \sum_{n=1}^{\infty} n^2b_n r^{n-1} \]
\[ \frac{1 - \sum_{n=2}^{\infty} na_n r^{n-1} + \sum_{n=1}^{\infty} nb_n r^{n-1}}{1 - \sum_{n=2}^{\infty} n^2a_n + \sum_{n=1}^{\infty} n^2b_n} \]

If \( \sum_{n=1}^{\infty} n^2a_n + \sum_{n=1}^{\infty} n^2b_n > 1 \), the numerator is negative for \( r \) sufficiently close to 1. Thus, there exists \( r \in (0, 1) \) for which the quotient is negative. Hence \( f \notin \mathcal{T}K_H \) and the proof is complete. \( \square \)

Corollary 2.5. If \( f \in \mathcal{T}H^* \), then \( f \) maps \( |z| < \frac{1}{2} \) onto a convex domain. The result is sharp, with extremal functions \( z - \frac{b}{2} + \frac{z^2}{2} \) and \( z - \frac{b}{2} - \frac{z^2}{2} \), \( 0 \leq b < 1 \).

Proof. It suffices to show that \( 2f(z) \in \mathcal{T}K_H \). From

\[ 2f \left( \frac{z}{2} \right) = z - \sum_{n=2}^{\infty} \frac{a_n}{2n-1} z^n - \sum_{n=1}^{\infty} \frac{b_n}{2n-1} \bar{z}^n, \]

it follows that

\[ b_1 + \sum_{n=2}^{\infty} n^2 \left( \frac{a_n}{2n-1} + \frac{b_n}{2n-1} \right) \leq b_1 + \sum_{n=2}^{\infty} n(a_n + b_n) \leq 1 \]

as needed. \( \square \)

For any compact family \( \mathcal{F} \), the maximum or minimum of the real part of any continuous linear functional occurs at one of the extreme points of the closed convex hull, \( \text{clco} \mathcal{F} \). We will use the necessary and sufficient coefficient conditions of Theorem 2.3 and its Corollary to determine the extreme points of the closed convex hulls of \( \mathcal{T}H^* \) and \( \mathcal{T}K_H \).
Theorem 2.6.
(a) Set $h_1(z) = z$, $h_n(z) = z - \frac{z^n}{n}$ $(n = 2, 3, \ldots)$, and $g_n(z) = z - \frac{z^n}{n^2}$ $(n = 1, 2, 3, \ldots)$. Then $f \in \text{clco } T_H^*$ if and only if

$$f(z) = \sum_{n=1}^{\infty} (\lambda_n h_n + \gamma_n g_n),$$

where $\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n - \sum_{n=1}^{\infty} \gamma_n$, $\lambda_j \geq 0$ and $\gamma_j \geq 0$. In particular, the extreme points of $T_H^*$ are $\{h_n\}$ and $\{g_n\}$.

(b) Set $h_1(z) = z$, $h_n(z) = z - \frac{z^n}{n}$ $(n = 2, 3, \ldots)$, and $g_n(z) = z - \frac{z^n}{n^2}$ $(n = 1, 2, 3, \ldots)$. Then $f \in \text{clco } TK_H$ if and only if

$$f(z) = \sum_{n=1}^{\infty} (\lambda_n h_n + \gamma_n g_n),$$

where $\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n - \sum_{n=1}^{\infty} \gamma_n$, $\lambda_j \geq 0$ and $\gamma_j \geq 0$.

Proof. We prove (a). Because the proof of (b) is similar, it is omitted. Suppose

$$f(z) = \sum_{n=1}^{\infty} (\lambda_n h_n + \gamma_n g_n) = z - \sum_{n=2}^{\infty} \frac{\lambda_n}{n} z^n - \sum_{n=1}^{\infty} \frac{\gamma_n}{n} z^n.$$

Then

$$\sum_{n=2}^{\infty} n \left(\frac{\lambda_n}{n}\right) + \sum_{n=1}^{\infty} n \left(\frac{\gamma_n}{n}\right) = 1 - \lambda_1 \leq 1,$$

and $f \in \text{clco } T_H^*$. Conversely, if $f \in \text{clco } T_H^*$ we set $\lambda_n = n a_n$ $(n = 2, 3, \ldots)$, $\gamma_n = n b_n$ $(n = 1, 2, 3, \ldots)$, and $\gamma_1 = 1 - \sum_{n=2}^{\infty} \lambda_n - \sum_{n=1}^{\infty} \gamma_n$. Then $f(z) = \sum_{n=1}^{\infty} (\lambda_n h_n + \gamma_n g_n)$ as needed. □

Remark 2.7. Most families investigated for univalent functions with negative coefficients are both convex and compact, including $T_H^{*0}$ and $TK_H^0$, so that the closed convex hulls are the families themselves. See [4] and [5]. This is not the case for the families in Theorem 2.6, since the extreme point $g_1(z) = z - \bar{z} \in \text{clco } T_H^* - T_H^*$. In [5], it is shown for $f \in T_H^{*0}$ and $|z| = r < 1$ that

$$r - \frac{r^2}{2} \leq |f(z)| \leq r + \frac{r^2}{2}.$$

We next show that the corresponding bounds for $T_H^*$ are quite different.

Theorem 2.8. If $f \in T_H^*$, then

$$\sup_{|z|=r} |f(z)| = 2r \quad \text{and} \quad \inf_{|z|=r} |f(z)| = 0.$$
Proof. In view of Theorem 2.3,

\[ |f(z)| \leq (1 + b_1)r + \sum_{n=2}^{\infty} (a_n + b_n) r^n \]
\[ \leq (1 + b_1)r + r^2 \sum_{n=2}^{\infty} (a_n + b_n) \]
\[ \leq (1 + b_1)r + \left( \frac{1 - b_1}{2} \right) r^2. \]

The upper bound follows upon noting the right side is an increasing function of \( b_1 \).

The lower bound is a consequence of the inequality

\[ |f(z)| \geq (1 - b_1)r - \left( \frac{1 - b_1}{2} \right) r^2. \]

To see that the result is sharp, we note that, for \( f(z) = z - b\overline{z} \), setting \( z = ir \) and \( z = r \) yields the upper bound and lower bound, respectively, upon letting \( b \to 1^- \).

\[ \square \]

Remark 2.9. Since \( z - b\overline{z} \in \mathcal{T}_K \), the same bounds remain valid for this subclass of \( \mathcal{T}_H^r \).

3. Fixed Coefficient Results

Denote by \( \mathcal{T}_H^r(b) \), \( 0 \leq b < 1 \), the subfamily of \( \mathcal{T}_H^r \) consisting of functions for which \( b_1 = b \) is fixed. It is easy to see that \( \mathcal{T}_H^r(b) \) is a compact and convex family for which the extreme points are

\[ h_1(z) = z - b\overline{z}, \quad h_n(z) = z - b\overline{z} - \left( \frac{1-b}{n} \right) z^n, \quad \text{and} \]
\[ g_n(z) = z - b\overline{z} - \left( \frac{1-b}{n} \right) z^n, \quad n = 2, 3, 4, \ldots. \]

It is of interest to determine bounds on the modulus as well as the real and imaginary parts of functions that are in \( \mathcal{T}_H^r(b) \). The results are unexpected.

Theorem 3.1. If \( f \in \mathcal{T}_H^r(b) \) and \( |z| = r \), then

\[ |f(z)| \geq r(1 - b) \left( 1 - \frac{r}{2} \right), \quad 0 \leq b < 1, \]

and, at least for \( 0.029 \leq b < 1 \)

\[ |f(z)| \leq \max \left\{ G(b, r), \ r(1 + b) + \frac{1-b}{3} r^3 \right\}, \quad (4) \]

where

\[ (G(b, r))^2 = \frac{3 + 3b + 2b^2}{3} r^2 + \frac{(1-b)^2}{4} r^4 \]

\[ + \frac{-8b^3 + \sqrt{(4b^2 + 3(1-b)^2(3+b)r^2)^3}}{27(1-b)^2}. \]
Proof. It suffices to consider the extreme points of $T^*_f(b)$. Since $(1-b)/n$ decreases with $n$, the lower bound clearly holds for $h_2(r) = g_2(r)$.

For $n \geq 3$,

$$|h_n(z)| \leq r(1 + b) + \frac{1-b}{n} r^n \leq r(1 + b) + \frac{1-b}{3} r^3 = |h_3(ir)|.$$ 

Similarly, $|g_n(z)| \leq |h_3(ir)|$ when $n \geq 3$. Thus, it suffices to compare $|h_3(ir)|$ with $\max_{\theta} |g_2(z)|$ and $\max_{\theta} |h_2(z)|$.

Straightforward, though messy, calculations reveal, for $|z| = r$ that

$$\max_{\theta} |h_2(z)|^2 = \frac{2 + 3b + 3b^2}{3} r^2 + \frac{(1-b)^2}{4} r^4$$

$$+ \frac{-8b^2 + \sqrt{b(4b + 3(1-b)^2(1+3b)r^2)}}{27b(1-b)^2}$$

for all $r$ when $\frac{3-\sqrt{6}}{3} < b < 1$ and for $0 < r < \frac{8b}{(1-b)(1-9b)}$ when $0 < b < \frac{3-\sqrt{6}}{3}$;

$$\max_{\theta} |g_2(z)|^2 = (G(b,r))^2$$

as defined above. Basic algebraic arguments lead to the conclusion that, at least for $0.02853 < b < 1$,

$$\max_{\theta} |h_2(z)|^2 < \max_{\theta} |g_2(z)|^2.$$ (5)

Thus, for $0.029 \leq b < 1$, we conclude that the maximum of $|f(z)|^2$, $f \in T^*_f(b)$ and $|z| = r$, is given by either $|h_3(ir)|^2$ or $(G(b,r))^2$. To see that both functions given on the right side of (4) can yield the maximum, note that when $b = 0.4$ we have that $\max_{\theta} |g_2(0.4)|^2 \approx 0.33061 > |h_3(0.4i)|^2 \approx 0.3281$ while $\max_{\theta} |g_2(0.8)|^2 \approx 1.4804 < |h_3(0.8i)|^2 \approx 1.4943$.

Remark 3.2. For $b = 0$, we know that $\max_{\theta} |h_2(z)| = \max_{\theta} |g_2(z)| = (r + \frac{r^2}{2})$; while, for $b \in (0,0.029]$, numerical considerations involving Maple, support the claim that inequality (5) holds for all $r \in (0,1)$. However, the bounding arguments used did not carry over to the whole range of $b$.

Remark 3.3. Applying basic algebraic arguments, it is easy to prove that, for fixed $b \geq 0.48$, $|h_3(ir)| > G_2(b,r)$ for all $r \in (0,1)$. Numerical considerations involving Maple indicate that, at least for each fixed $b$ with $0 < b < .3$, $|h_3(ir)| < G_2(b,r)$ for $0 < r < 1$.

The bounds on $\Re f(z)$ must occur at an extreme point. Note that $\Re h_1(z) = z - bz = r(1-b) \cos \theta$ and, for $n \geq 2$,

$$\Re h_n(z) = \Re g_n(z) = r(1-b) \left[ \cos \theta - \frac{r^{n-1}}{n} \cos n\theta \right].$$

We have $\Re h_n(z) \geq h_2(-r) = -r(1-b)(1 + \frac{r}{2})$.

The upper bound is much more surprising. Observe that $\Re h_1(z) \leq h_1(r) = r(1-b)$. Thus for $\Re h_n(z)$ to be maximal for $n \geq 2$ and some $r$, there must be a $\theta$ for which $t_n(r, \theta) = \cos \theta - \frac{r^{n-1}}{n} \cos n\theta > 1$. 
We first look at the boundary \( r = 1 \). Then \( t_n(1, \theta) = t_n(\theta) = \cos \theta - \frac{\cos n \theta}{n} \). If \( t_n(\theta) > 1 \), we have \( \cos \theta > 0 \) and \( \cos n \theta < 0 \). Now \( t'_n(\theta) = -\sin \theta + \sin n \theta = 0 \) when \( \sin \theta = \sin n \theta \), and hence \( \cos \theta = -\cos n \theta = \cos(\pi - n \theta) > 0 \). Thus, for each \( n \), \( t_n(\theta) \leq t_n\left(\frac{\pi}{n+1}\right) = \frac{n+1}{n} \cos\left(\frac{\pi}{n+1}\right) \). Numerical considerations lead to the conclusion that \( t_n\left(\frac{\pi}{n+1}\right) \) increases for \( n \leq 8 \) and then decreases with \( n \). Hence,

\[
t_n(\theta) \leq t_8\left(\frac{\pi}{9}\right) = \frac{9}{8} \cos\left(\frac{\pi}{9}\right) \approx 1.057.
\]

In particular, for \( f \in T_H^*(b) \), \( \Re f(z) \) attains its maximum when \( |z| = r \) is close to 1 for \( h_8(z) \). Moreover, max \( t_n(r, \theta) \leq \max t_n(1, \theta) \leq t_8(1, \frac{\pi}{9}) \approx 1.057 \). This appears to indicate that \( t_n(r, \theta) < h_1(r) = r(1-b) \).

These observations are summarized in the following theorem.

**Theorem 3.4.** If \( f \in T_H^*(b) \), then for \( |z| = r \) we have \( \Re f(z) \geq -r(1-b)(1+\frac{r}{2}) = h_2(-r) \). In addition, \( \Re f(z) \leq r(1-b) = h_1(r) \) for \( 0 < r \leq r_0 \approx 0.900 \); \( \Re f(z) \) attains its upper bound with \( h_0(z) \) for \( r_0 < r \leq r_1 \approx 0.915 \) and with \( h_8(z) \) for \( r_1 < r < 1 \).

For the bounds on \( \Im f \) over \( f \in T_H^*(b) \), again restricting ourselves to the extreme points of \( T_H^*(b) \), we note that \( \Im h_1(\rho e^{i\theta}) = (1+b)r \sin \theta \), \( \Im h_n(\rho e^{i\theta}) = (1+b)r \sin \theta - \frac{1-b}{n} r^n \sin n \theta \), and

\[
s_n(r, \theta) := \Im g_n(\rho e^{i\theta}) = (1+b)r \sin \theta + \frac{1-b}{n} r^n \sin n \theta.
\]

We have that \( \Im h_n(\rho e^{i\theta}) \) and \( \Im g_n(\rho e^{i\theta}) \) assume the same set of values for \( -\pi < \theta \leq \pi \). Furthermore,

\[
\max_\theta \{ \Im g_3(\rho e^{i\theta}) \} = s_3\left(r, \frac{\pi}{2}\right) = (1+b)r + \frac{1-b}{3} r^3 > \Im h_1(\rho i r)
\]

and

\[
\min_\theta \{ \Im g_3(\rho e^{i\theta}) \} = s_3\left(r, -\frac{\pi}{2}\right) = -(1+b)r - \frac{1-b}{3} r^3 < \Im h_1(\rho (-ir)).
\]

It follows that finding bounds for \( \Im f \), \( f \in T_H^*(b) \), is equivalent to comparing \( s_3(r, \frac{\pi}{2}) \) and \( s_3(r, -\frac{\pi}{2}) \) with \( \max_\theta \{ \Im g_2(\rho e^{i\theta}) \} \) and \( \min_\theta \{ \Im g_2(\rho e^{i\theta}) \} \), respectively. The results of the comparison are summarized in the following.

**Theorem 3.5.** For fixed \( b \), \( 0.2 < b < 1 \), and \( f \in T_H^*(b) \),

\[
-(1+b)r + \frac{1-b}{3} r^3 \leq \Im f(\rho e^{i\theta}) \leq (1+b)r + \frac{1-b}{3} r^3, \text{ for } |z| = r. \quad (6)
\]

The result is sharp.

**Proof.** For fixed \( b \) and \( r \), the extrema for \( \Im g_2(\rho e^{i\theta}) = (1+b)r \sin \theta + \frac{1-b}{2} r^2 \sin 2 \theta \) occur when \( \cos \theta^* = \frac{-(1+b)+m(b,r)}{4(1-b)r} \) where \( m(b, r) = \sqrt{(1+b)^2 + 8(1-b)^2 r^2} \). Then the minimum and maximum are

\[
s_2(r, -\theta^*) = -H(r, b) \quad \text{and} \quad s_2(r, \theta^*) = H(r, b),
\]

respectively, where

\[
H(b, r) = \frac{(3(1+b) + m(b, r))\sqrt{8(1-b)^2r^2 - 2(1+b)^2 + 2(1+b)m(b, r)}}{16(1-b)}.
\]
Finally, we note that

\[
(3(1 + b) + m(b, r)) \sqrt{8(1 - b)^2 r^2 - 2(1 + b)^2 + 2(1 + b)m(b, r)} \\
\leq 16(1 - b) \left( (1 + b)r \sin \theta + \frac{1 - b}{3} r^3 \right)
\]

at least for \( b \geq 0.2 \). Consequently, for \( b \geq 0.2 \), the sharp bounds for \( \text{Im} f \) are given by \( g_3 \).

Remark 3.6. Note that, for \( b \in [0, 0.2) \), the maximum and minimum of \( \text{Im} f \), \( f \in T_{H^*}(b) \), are given by \( g_2 \) for some \( r \). For example, \( H(0.1, 0.25) \approx 0.28048 > g_3(0.1, 0.25) \approx 0.27969 \) while \( H(0.1, 0.75) \approx 0.94179 < g_3(0.1, 0.75) \approx 0.95156 \).

References


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