SPLITTING OF ACOUSTIC RESONANCES IN
DOMAINS CONNECTED BY A THIN CHANNEL

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Abstract. For two-dimensional domains connected by a thin channel the perturba­
tion of eigenvalues and resonances are considered. The techniques used are based on
integral equations, containing contracting operators.

The calculation of shifts and splittings of resonances and eigenvalues for weakly-
connected domains is rather a classical problem of mathematical physics. We men­
tion here only the latest publications, concerning this question for a homogenous
Dirichlet boundary condition, [1 - 6]. All of them give realistic estimates for cor­
responding shifts and splittings, but the proper calculation of them or even lower
estimates for shifts and splittings meets serious obstacles.

It appears, that the straightforward approach to the calculation of shifts and
splittings of resonances and eigenvalues, suggested in a more simple problem for
two domains connected by a small small opening (see [7]) can be used to calculate
all spectral characteristics of the acoustic problem described above, if accomplished
with asymptotic estimates for Green functions, based on the iterated Hilbert iden­
tity. The corresponding integral equations can be written directly in terms of Green
functions of unperturbed problems and the channel. The integral equations, which
appear in this way, contain an integral operator, which is generally represented in
every bounded domain of the complex plane of the spectral parameter λ as a sum
of contracting operator and a finite dimensional rational function of λ. The corre­
sponding equations can be solved easily by combination of iterations and matrix
algebra methods. In what follows, we suppose that the boundaries of the nonper­
turbed domain Ω₁ ∪ Ω₃ and of the channel are smooth, and have inner angles at
the points of contact.

1. Gotlib's Type Integral Equations for a Thin Channel

Let Ω₁ = Ω_in , Ω₃ = Ω_out be inner and outer domains, joined by a thin
"ε-channel" Ω₂ = Ω₂, which is formed by two parallel curves l₀ , l₁ , dist(l₀, l₁) = ε,
ε ≤ 1. We assume, that the channel Ω₂ meets ∂Ω₁ and ∂Ω₃ in an orthogonal way
at the small flat pieces γ₁, γ₃ of ∂Ω₁ and ∂Ω₃. The Laplace operator we consider
here is defined as Friedrichs's extension of −Δ in L₂(Ω) ≡ L₂(Ω₁ ∪ Ω₂ ∪ Ω₃). It
means,that the homogenous Dirichlet boundary condition at the common part
of boundaries ∂Ω₁ ∩ ∂Ω₂ = γ₁, ∂Ω₂ ∩ ∂Ω₃ = γ₃ is replaced by the "non-jump"
conditions for uᵢ = u|Ωᵢ:

\[ \frac{\partial u_{1,3}}{\partial n} \bigg|_{γ₁,3} - \frac{\partial u_2}{\partial n} \bigg|_{γ₁,3} = 0; \]

\[ u|_{∂Ω\setminus(γ₁∪γ₂)} = 0. \]

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The domain of the corresponding Laplace operator is contained in the Sobolev class $W^{1,0}_2(\Omega)$ of all functions which possess square integrable first order derivatives and vanish on $\partial \Omega$; thus Meixner conditions at the inner angles between $\partial \Omega_1$, $\partial \Omega_3$ and $\partial \Omega_2$ are fulfilled automatically. From the very beginning we suppose, that all spectral properties of the Laplacians $\Delta_i$ in $L^2(\Omega_i)$ with homogeneous boundary conditions on $\partial \Omega_i$ are known. Note, that the spectral properties of Laplacian $\Delta_2$ in the thin channel can be calculated by a perturbation procedure, if $\varepsilon$ is small enough. In the two dimensional case, which we discuss here, they can be deduced from the integral equations of Lippmann-Schwinger type, containing the derivative of the holomorphic function, which maps the curved channel $\Omega_2$ onto the corresponding direct channel.

We consider two cases:

1. $\Omega_3$ is a bounded domain.

2. $\Omega_3$ is an unbounded domain with a bounded border $\partial \Omega_3$ ($\Omega_3$ contains a neighborhood of $\infty$).

Both $\Omega_1$, $\Omega_2$ are supposed to be bounded. In both cases the integral equation for the resolvent kernel can be deduced for any value of the spectral parameter. But in the second case we can deduce for real values of the spectral parameter $\lambda$ the inhomogeneous equation for scattering waves as well.

Let us denote by $K^\varepsilon_i$, $K^\varepsilon$ two discs, centered at $x_i^1$, $x_0^3$, in the middle of $\gamma_1$, $\gamma_3$, $\text{diam } K^\varepsilon_i = |\gamma_i| = \varepsilon$, $i = 1, 3$, $\partial K^\varepsilon_i = \Sigma_{1i} \cup \Sigma_{12}$, $\Sigma_{1s} = \partial K_1 \cap \Omega_s$, $s = 1, 2$; $\partial K^\varepsilon = \Sigma_{32} \cup \Sigma_{33}$, $\Sigma_{3s} = \partial K_3 \cap \Omega_s$, $s = 2, 3$.

In what follows we omit the parameter $\varepsilon$ in the notations. We construct the integral equation for the "compensating" term $g_0$ of the Green function in $\Omega$

$$G(x, y) = G_0(x, y) + g_0(x, y),$$

which satisfies the inhomogeneous boundary conditions

$$\begin{cases} 
-\Delta g = k^2 g, \\
\left.g_0(x, y)\right|_{x \in \partial \Omega} = -\left.G_0(x, y)\right|_{x \in \partial \Omega}.
\end{cases}$$

Here $G_0$ is the singular solution of the Helmholtz equation in $R_2$,

$$G_0(|x - y|) = \frac{i}{4} H_0^1(k|x - y|).$$

Actually we derive the integral equation for the pair \{g_1, g_2\}, $g_1 = g|_{\gamma_1}$, $g_3 = g|_{\gamma_3}$. The contracting property of the corresponding integral operator we get as a corollary of results of Sections 2, 3, 4. Now we deduce this equations just formally.

If $g_1$, $g_2$ are known, then the values of $g$ on can be calculated as the convolution of

$$g_1|_{\gamma_1} + (-G_0)|_{\partial \Omega_1 \setminus \gamma_1}$$

with the Poisson kernel of the Helmholtz equation in $\Omega_1$:

$$P_1(x, s) \equiv \frac{\partial G_1}{\partial n_s}(x, s).$$
Here $G_1$ is the Green-function of the Laplacian in $\Omega_1$ with the homogeneous Dirichlet boundary condition at the boundary $\partial \Omega_1$, $y \in \Omega_1$:

$$g(x,y)|_{x \in \Sigma_1} = \left[ \int_{\gamma_1} P_1(x, s) g_1(s) ds + \int_{\partial \Omega_1 \setminus \gamma_1} P_1(x, s)[-G_0(s, y)] ds \right]_{x \in \Sigma_1}. $$

Similarly we get for domains $\Omega_3$, $\Omega_2$:

$$g(x,y)|_{x \in \Sigma_{33}} = \int_{\gamma_3} P_3(x, s) g_3(s) ds, \quad (1)$$

$$g(x,y)|_{x \in \Sigma_{12}} = \left[ \int_{\gamma_1} P_2(x, s) g_1(s) ds + \int_{\gamma_3} P_2(x, s) g_3(s) ds \right]_{x \in \Sigma_{12}}, \quad (2)$$

$$g(x,y)|_{x \in \Sigma_{32}} = \left[ \int_{\gamma_1} P_2(x, s) g_1(s) ds + \int_{\gamma_3} P_2(x, s) g_3(s) ds \right]_{x \in \Sigma_{32}}. \quad (3)$$

Here $P_2$, $P_3$ are Poisson kernels in $\Omega_2$, $\Omega_3$. If $y \in \Omega_3$, then an inhomogeneous term, similar to the second addenda in the right part of $g(x,y)|_{x \in \Sigma_1}$, appears in the representation formula (1).

Considering the reduced values of $g$ onto $\gamma_1, \gamma_3$ we can calculate $g_1, g_3$ in form of convolutions with the Poisson kernels $P_{1,3}^0$ of the Helmholtz equation in the discs $K_1$:

$$g_1(x,y)|_{x \in \gamma_1} = \left\{ \int_{\Sigma_{11}} P_{1}^0(x, s) g(s, y) ds + \int_{\Sigma_{21}} P_{2}^0(x, s) g(s, x) ds \right\} \right|_{x \in \gamma_1}, \quad (4)$$

and $K_3$

$$g_3(x,y)|_{x \in \gamma_3} = \left\{ \int_{\Sigma_{32}} P_{1}^0(x, s) g(s, y) ds + \int_{\Sigma_{33}} P_{1}^0(x, s) g(s, x) ds \right\} \right|_{x \in \gamma_3}. \quad (5)$$

Combination of the equations (1 – 5) gives a system of integral equations for $g_1$, $g_3$. This system can be written in the form

$$\begin{pmatrix} g_1 \\ g_3 \end{pmatrix} = \mathcal{K} \begin{pmatrix} g_1 \\ g_3 \end{pmatrix} + \mathcal{F}, \quad (6)$$

where $\mathcal{F}$ is an inhomogeneous term, defined by the position of $y$, and $\mathcal{K}$ is an integral operator, which proves to be contracting one in the Banach space of all continuous two-component vector functions, defined onto $\gamma_1, \gamma_3$:

$$C_\gamma = C(\gamma_1) + C(\gamma_3),$$

for the values of the spectral parameter $k$ not equal to the eigenvalues of inner or outer problem and the width $\varepsilon$ of the channel $\Omega_2$ small enough.
The case \( y \in \Omega_3 \) can be treated in a similar way. Generally the integral equations for the perturbed scattered waves in the unbounded domain can be derived the same way with the only difference that the nonperturbed Green function \( G_0(x,y) \) should be replaced by the nonperturbed scattered waves in \( \Omega_3 \) or just by the exponentials \( e^{-ik(x,v)} \) (if \( \Omega_3 \) contains some neighborhood of \( \infty \)).

Formally in the last case the corresponding equation can be deduced from (6) by using the asymptotics of the Green function at infinity in the direction \( \nu \):

\[
G_0(x,y) \simeq e^{-ik(x,v)}G_0(0,y), \quad y \to \nu \infty,
\]

Actually it contains the same contracting integral operator \( K \). In the case when the value of the spectral parameter is singular for one of the domains \( (\Omega_i) \), the integral equations need some regularization, which includes explicit substraction of singular terms (see below).

2. Estimates of Green Functions and Poisson Kernels of the Helmholtz Equations in \( \Omega_1, \Omega_2 \)

**Lemma 1.** The Green function \( G_1(x,y,\lambda) \) of the homogeneous Dirichlet problem for the Helmholtz equation in \( \Omega_1, -\Delta u = \lambda u \), admits the representation in the following form for \( \lambda \) approaching the isolated eigenvalue \( \lambda_1 \) of the corresponding operator

\[
G_\lambda(x,y) = G_{-1}(x,y) + \sum_{s=1}^{N_1} \frac{\varphi_{is}(x)\varphi_{is}(y)}{\lambda_1 - \lambda} + \tilde{g}(x,y),
\]

where \( \{\varphi_{is}\} \) is the complete orthogonal system of eigenvectors of the problem

\[-\Delta \varphi = \lambda \varphi\]

in the corresponding eigenspace \( N_1 = \text{Ker}(-\Delta - \lambda_1 I) \), and \( \tilde{g}(x,y,\lambda) \) is a smooth function, which admits uniform estimates, when \( x \to y, \lambda \to \lambda_1 \).

**Proof.** The proof is based on the iterated Hilbert identity which has the following form:

\[
G_\lambda = G_{-1} + \sum_{s=1}^{N} (\lambda + 1)^s G^s_{-1} + (\lambda + 1)^{N+1} G^N_{-1} G_\lambda,
\]

\( G^s_{-1} = G_{-1} \times G_{-1} \times \ldots G_{-1} \) (s-times). Using the spectral representation for \( G_\lambda \), we can write the last term in the right part of (9) as a sum of a singular and regular terms

\[
G_1^{N+1} G_\lambda = \sum_{\lambda_s \neq \lambda_1} \frac{p_s}{(\lambda_s + 1)^{N+1}(\lambda_s - \lambda)} + \frac{p_1}{(\lambda_1 + 1)^{N+1}(\lambda_1 - \lambda)};
\]
here $\mathcal{P}_s$ are the orthogonal projectors onto the eigenspace $\mathcal{N}_s$, corresponding to the eigenvalues $\lambda_s$ of the Laplacian in $\Omega_1$. Using the fact, that even the first iteration of the resolvent kernel of the Laplacian in the two-dimensional case gives a continuous function, and the following iterations give smooth functions, we see, that the first addenda in the right part of (9) is an integral operator with the smooth kernel in $\Omega_1 \times \Omega_1$, provided the eigenvalue $\lambda_1$ is isolated. Thus we have the representation:

$$G_{\lambda}(x, y) = G_{-1}(x, y) + \left( \frac{\lambda + 1}{\lambda_1 + 1} \right)^{N+1} \frac{\sum_{t=1}^{N} \varphi_{1t}(x) \varphi_{1t}(y)}{\lambda_1 - \lambda}$$

$$+ \sum_{\lambda_p \neq \lambda_1}^{N_p} \frac{\sum_{r=1}^{N_p} \varphi_{rp}(x) \varphi_{rp}(y)}{\lambda_p - \lambda} \left( \frac{\lambda + 1}{\lambda_p + 1} \right)^N$$

$$+ \sum_{1}^{N} (\lambda + 1)^{s} G_{-1}^{s+1}(x, y).$$  \hspace{1cm} (10)

The third and the fourth members in the right side of the last equation are continuous functions in $\Omega_1 \times \Omega_2$, hence

$$G_{\lambda}(x, y) = G_{-1}(x, y) + \left( \frac{\lambda + 1}{\lambda_1 + 1} \right)^{N+1} \frac{\sum_{t=1}^{N} \varphi_{1t}(x) \varphi_{1t}(y)}{\lambda_1 - \lambda}$$

$$+ \sum_{1}^{N} (\lambda + 1)^s G_{-1}^{s+1}(x, y) + O(1),$$

where $O(1)$ is bounded uniformly in $\Omega_1 \times \Omega_1$, when $\lambda$ is in some neighbourhood of the isolated eigenvalue $\lambda_1$, which does not contain other eigenvalues. \[\] \hspace{1cm} Remark. The last statement is true actually for any regular differential operator, for which the series of some power $N$, $N > 0$ of eigenvalues is convergent

$$\sum_{\lambda_p \neq 0} |\lambda_p|^{-N} < \infty,$$

and the eigenfunctions are smooth functions. The precise formulations and proofs of relevant results will be published elsewhere.

Corollary. A similar representation holds for the Poisson kernel of the Helmholtz equation:

$$\mathcal{P}_1 \sim \frac{\partial G_{\lambda}(x, y)}{\partial n_x} \bigg|_{x \in \partial \Omega_1} = \frac{\partial G_{-1}(x, y)}{\partial n_x} + \sum_{1}^{N} (\lambda + 1)^s \frac{\partial G_{-1}(x, y)}{\partial n_x} \times G_{-1}^{s}(x, y)$$

$$+ \left( \frac{\lambda + 1}{\lambda_1 + 1} \right)^{N+1} \frac{1}{\lambda_1 - \lambda} \sum_{r=1}^{N_1} \frac{\partial \varphi_{1r}(x)}{\partial n_x} \varphi_{1r}(y)$$

$$+ O(1).$$  \hspace{1cm} (11)
In the case, when $\Omega_3$ is a bounded domain, the same sort of asymptotic estimates can be written in $\Omega_3$. The case of an unbounded domain $\Omega_3$ containing a neighborhood of infinity is described by the following:

**Lemma 2.** Let us consider the Helmholtz equation in $\Omega_3$, with the homogeneous Dirichlet conditions at the boundary. Let us denote by $\psi_\nu$ the scattered waves of the exterior problem for initiated by the plane wave $e^{ik(\nu,x)}$, incoming from infinity along the direction $\nu$.

Then the behaviour of the Green function of the Helmholtz equation in $\Omega_3$, for $\lambda$ approaching $\lambda_0 \pm i\alpha$, $\lambda_0 > 0$ is defined by the following formula

$$G_\lambda(x,y) = G_{-1}(x,y) \pm \frac{i}{4} \int_{\Sigma_1} \psi_\nu(x,\lambda_0) \overline{\psi_\nu(y,\lambda_0)} d\nu +$$

$$\left\{ \sum_{1}^{N} (\lambda + 1)^{s} G_{-1}^{s+1}(x,y) + v.p. \int_{(\lambda_0)}^{z} \frac{dz}{8\pi^2} \int_{\Sigma_1} d\nu \left( \frac{\lambda + 1}{\lambda + 1} \right)^{N+1} \frac{\psi_\nu(x,z) \overline{\psi_\nu(y,z)}}{z - \lambda_0} \right\} . \quad (12)$$

Here $g_d$ is the contribution, which comes from the discrete spectrum of the problem, and $\int_{(\lambda_0)}$ is the integral in a sense of "principal value" over $[0, \infty)$ with singularity at $\lambda_0$, $\lambda_0 > 0$. The members, which are staying in the bracket admit the uniform estimate in some neighborhood of any "generic" point of the absolutely continuous spectrum of the unperturbed Laplacian, where the scattered waves are smooth functions of the spectral parameter.

The proof will be published elswere. Actually it follows the same idea of using iterated Hilbert identity, as the previous one (Lemma 1, concerning the nonsingular case) with the only difference, that now the standard formula for the boundary values of the Cauchy-type integral must be used. The statement of the Lemma 2 holds for Schrödinger operators in the exterior domain with the real integrable potentials potentials, and can be generalized for quasiconical domains as well.

**Corollary.** In the generic case when the scattered waves of the nonperturbed spectral problem are smooth near the real positive point $\lambda$ of the spectral parameter, then the Poisson kernel admits the following representation

$$P_3(x,y) = \left. \frac{\partial G_\lambda(x,y)}{\partial n_x} \right|_{x \in \partial \Omega_3} = \frac{\partial G_{-1}(x,y)}{\partial n_x} \pm \frac{i}{4} \int_{\Sigma_1} \left( \frac{\partial \psi_\nu(x,\lambda_0)}{\partial n_x} \right) \overline{\psi_\nu(y,\lambda_0)} d\nu + O(1).$$

### 3. "Acoustic Conductivity" of the Thin Channel

Note first, that scattering problems for curved channels were studied intensely last years, see [8] and references there. We discuss here the case of the thin channel only. This case is selected by the condition, that the spectral parameter $\lambda$ is chosen below the spectral treshold of the channel.
Acoustic conductivity of a thin channel is defined by the behaviour of the solution of the Dirichlet problem in it for the Laplacian with a nonvanishing boundary data only on one of the end sections $\gamma_1$, or $\gamma_2$, for instance:

$$-\Delta u_1 = \lambda u_1,$$

$$u_1|_{\gamma_1} = \varphi_1 u_1|_{\partial \Omega_2 \setminus \gamma_1} = 0.$$  

The simple method of estimating “conductivity” of a thin channel is based on maximum principle for elliptic equations, see [9], where A.M. Iljin applies it to the transformed elliptic equation. In our case this approach looks as follows.

Supposing, that the equation of the border of the channel $l_1$ is given in the natural form

$$\frac{d^2 x}{ds^2} = \frac{1}{\rho(s)} n(s),$$

where $s$ is the length of arc $l_1$ between $x_0$ and $x$, and $n(s)$ is the normal to $l_1$ at the point $x$, which forms a positive frame with the tangent vector, $t = \frac{dx}{ds}$

$$(t, n)^+,$$

we can pass to the coordinates $(s, h)$, connected with this frame and corresponding to the metric form

$$\left(1 - \frac{h}{\rho}\right)^2 ds^2 + dh^2.$$

The Laplacian in the channel $\Omega_2 = \{0 < s < L, 0 < h < \epsilon\}$ looks now like

$$\Delta^* = \frac{1}{1 - \frac{h}{\rho} \frac{d}{ds}} \left[ \frac{1}{1 - \frac{h}{\rho} \frac{d}{ds}} \Delta^* \right] + \frac{1}{1 - \frac{h}{\rho} \frac{d}{dh}} \left[ \left(1 - \frac{h}{\rho}\right) \frac{d}{dh} \right].$$

Considering an enlarged channel $\Omega_2^{\delta} = \{0 < s < L, -\delta < h < \epsilon + \delta\}$, we define the function

$$\psi_0 = \sin \pi \left\{ \ln \left( \frac{1 - \frac{h}{\rho}}{1 + \frac{\delta}{\rho}} \right) \right\} \sinh k \left\{ (L - s) + h \int_s^L \frac{dt}{\rho} \right\},$$

which is obviously positive in $\Omega_2$ and vanishing on the part of the boundary $\Omega_2^{\delta}$, not containing $\gamma_1$. For small $\epsilon, \delta$ we have

$$\frac{1}{\psi_0} \Delta \psi_0 \approx -\frac{\pi^2}{\ln \left( \frac{1 - (\epsilon + \delta) \rho^{-1}}{1 + \delta \rho^{-1}} \right)} \frac{1}{\rho^2} + k^2 \approx k^2 - \frac{\pi^2}{(\epsilon + 2\delta)^2}.$$  

Replacing $u = u_1$ by $v \psi_0$, we get for $v$ the following equation

$$-\Delta v - \lambda v - \frac{1}{\psi_0} \Delta \psi_0 v - \frac{2}{\psi_0 \left(1 - \frac{h}{\rho}\right)^2} \frac{dv}{ds} \frac{d\psi_0}{ds} - \frac{2}{\psi_0} \frac{dv}{dh} \frac{d\psi_0}{dh} = 0$$

$$v|_{\partial \Omega_2 \setminus \gamma_1} = 0, \quad v|_{\gamma_1} = \psi_0^{-1} \varphi_1.$$
Using the fact, that \( \frac{1}{\psi_0} \Delta \psi_0 \) is a large negative number for thin channel, we see, that the maximum principle for the solutions of the last equation is fulfilled for any fixed real \( \lambda \) if \( \varepsilon, \delta \) are small enough, \( 0 < \varepsilon + 2\delta < 1 \), since
\[
\frac{1}{\psi_0} \Delta \psi_0 + \lambda = \lambda + k^2 - \frac{\pi^2}{(\varepsilon + 2\delta)^2} < -1.
\]
Thus we get
\[
|u| \leq \max |v|\psi_0 = \max |\varphi_1| \sin^{-1} \pi \left\{ \frac{\ln \left( \frac{1 - \frac{h}{2}}{1 + \frac{h}{2}} \right)}{\ln \left( \frac{1 - \frac{\varepsilon + \delta}{2}}{1 + \frac{\varepsilon + \delta}{2}} \right)} \right\}_1 \times \sin \pi \left\{ \frac{\ln \left( 1 - \frac{\varepsilon + \delta}{1 + \frac{\varepsilon + \delta}{2}} \right)}{\ln \left( 1 - \frac{\varepsilon + \delta}{1 + \frac{\varepsilon + \delta}{2}} \right)} \right\}_1 \times \sinh k \left\{ L - s + h \int_0^s \frac{dt}{\rho} \right\} \sinh k \left\{ L + h \int_0^t \frac{dt}{\rho} \right\} \leq \max |\varphi_1| \text{Const} e^{-k \{ s - h \int_0^s \frac{dt}{\rho} \}}, \quad 0 < s < L. \quad (13)
\]
Here Const is defined by the first factor of \( \psi_0 \) inside \( \Omega_2 \).

This exponential estimate for the solution of the Dirichlet problem \( u = u_1 \) serves to prove the exponential decay of splittings and shifts of resonances for narrow channels. In particular it follows from the estimate above, that the Poisson kernel \( P_{x,y} \) decreases exponentially if \( x \in \gamma_1 \) and \( y \) is running along the channel to \( \gamma_2 \) (or vice versa):
\[
|\mathcal{P}_\lambda(x, y)| \leq \text{Const} e^{-k_x \text{dist}(x, y)},
\]
where \( k_x > \frac{\pi}{2} - \lambda > 0 \) and the distance between \( x, y \) is measured along the channel. This exponential estimate for the Poisson kernel shows, that the matrix elements of the integral operator \( \mathcal{K} \) in the equation (6), which correspond to the transmittance of excitations from \( \gamma_1 \) to \( \gamma_2 \) and vice versa, are exponentially small in a sense of the \( C(\gamma) \) norm. Together with the compressing property, which will be verified in the next section, it means, that the perturbations of all eigenvalues of the auxiliary problems in \( \Omega_1 \cup \Omega_2 \) and in \( \Omega_2 \cup \Omega_3 \) (see below) produced by the opening the opposite end of the channel \( \Omega_2 \) are exponentially small.

But the evaluation of these perturbations needs more information. Let us denote by \( f_\varepsilon \) the function, which maps conformally the direct channel \( \Omega_2^0 = \{ \omega : 0 \leq \Re \omega \leq L, 0 \leq \Im \omega \leq \varepsilon \} \) onto \( \Omega_2 \). Supposing this function is known, we can replace the Dirichlet problem for \( u = u_1 \) in \( \Omega_2 \) by the Dirichlet problem for the conformally transformed problem in the direct channel: \( \tilde{u} = u \circ f_\varepsilon \) in the direct channel \( \Omega_2^0 \)
\[
-\Delta u - \lambda u = -\frac{1}{|f_\varepsilon'(\omega)|^2} \Delta_{\omega} \tilde{u} - \lambda \tilde{u} = 0.
\]
Then the Green function for the perturbed problem can be found as a solution of the corresponding Lippman-Schwinger equation
\[
\tilde{G} = \tilde{G}_0 - \lambda \int_{\Omega_2^0} \tilde{G}_0 |f_\varepsilon'|^2 \mathcal{G} d^2 \omega.
\]
Investigating this equation by standard methods, we get the following statement.
Lemma 3. For any $\lambda$ and $\varepsilon$ small enough the Green function $\tilde{G}$ of the transformed problem and the corresponding Poisson kernel $\tilde{P}$ satisfies the following asymptotical conditions:

$$
\tilde{G} \sim \tilde{G}_0 + O\left(\frac{\varepsilon^2}{\min \rho^2}\right),
$$

$$
\tilde{P} \sim \tilde{P}_0 + O\left(\frac{\varepsilon^2}{\min \rho^2}\right).
$$

In particular the leading exponential term of the Poisson kernel $P(0, h'; s, h)$, $s \sim L - \varepsilon$, $\varepsilon/\rho \ll 1$, is given by the expression

$$
\frac{1}{\varepsilon} \sin \frac{\pi h}{\varepsilon} \sin \frac{\pi h'}{\varepsilon} \sinh \frac{\sqrt{\pi^2/\varepsilon^2 - \lambda(L - s)}}{\sinh \sqrt{\pi^2/\varepsilon^2 - \lambda L}}.
$$

4. Auxiliary Problems and Estimates of Integral Operators

The integral operator, which appeared in equation (6) is constructed of Poisson kernels in matrix form with elements, which are bounded operators in the space $C(\gamma_1) + C(\gamma_3)$:

$$
\mathcal{K} = \left\{ P_1^0(P_1 + P_2), P_3^0 P_2, P_3^0(P_3 + P_2) \right\}.
$$

The diagonal terms have a special sense, being connected with two auxiliary problems, which should be discussed before solving the central problems about splitting resonances and eigenvalues. The auxiliary problems are the following:

(i) Problem of attaching of the thin channel $\Omega_2$ to the inner domain $\Omega_1$. It is the spectral Dirichlet problem in $\Omega_1 \cup \Omega_2$ with the homogeneous Dirichlet boundary condition on $\partial(\Omega_1 \cup \Omega_2) = (\partial \Omega_1 \cup \partial \Omega_2) \setminus \gamma_1$.

(ii) Problem of attaching the thin channel $\Omega_2$ to the outer domain $\Omega_3$. It is the spectral Dirichlet problem in $\Omega_2 \cup \Omega_3$ with the homogeneous Dirichlet boundary condition on $\partial(\Omega_2 \cup \Omega_3) = (\partial \Omega_2 \cup \partial \Omega_3) \setminus \gamma_3$.

The terms staying in (14) outside of the diagonal describe the interaction between these problems. We suppose, that the eigenfunctions of the outer problem are smoothly dependent of the spectral parameter, so we discuss only generic case now. At the moment we do not have a proper description of conditions, which guarantee the generic case for a given value $\lambda$, corresponding to the outer (or auxiliary outer II) problem.

The analysis of auxiliary spectral problems is described in Appendix I and in Appendix II. Here base on this material to describe solutions of two main problems.
I. Let us suppose, that the auxiliary spectral problem is solved in \( \Omega_1 \cup \Omega_2 \), and we have a group of eigenvalues \( \{ \lambda_s \} \) of it, which are separated from the rest of the spectrum \( \{ \lambda_p \} \), \( \text{dist}[\{\lambda_s\}, \{\lambda_p\}] \geq \delta_\varepsilon > 0 \). We suppose, that the separation of eigenvalues of the auxiliary inner problem takes place, if for decreasing \( \varepsilon \) the variable \( \delta_\varepsilon \) has a lower bound

\[
\delta_\varepsilon \geq \delta_0, \quad \varepsilon \to 0.
\]

We call this group \( \{ \lambda_s \} \) \emph{asymptotically separated}. Practically it means, that the group \( \{ \lambda_s \} \) is formed by a family of "genetically" close eigenvalues of the unperturbed inner problem, which is separated from the rest of the spectrum by the distance \( \sim \delta_0 \). We suppose also, that no resonance states of the outer problem are situated in the neighbourhood of the group. Based on the accomplished spectral analysis of the inner perturbed spectral problem, we can study the splitting of resonances, caused by the removing of the wall \( \gamma_3 \). The Laplace operator in \( \Omega_1 \cup \Omega_2 \cup \Omega_3 \) with no walls \( \gamma_1, \gamma_3 \) we call just the "perturbed operator".

**Theorem 1.** If there exists a group of eigenvalues of the auxiliary inner problem, which is asymptotically separated from the rest of the spectrum of the auxiliary inner problem and from the resonances of the outer problem, then removing the wall \( \gamma_3 \) causes the appearance a series of resonances (or, possible, eigenvalues) of the perturbed operator which can be calculated from the integral equation (6). The total number of them is equal to the total multiplicity of the eigenvalues in the separated group.

**Proof.** Suppose that the spectral analysis of the auxiliary inner problem is accomplished, and we have a resolvent kernel \( G^{\text{in}}_{\lambda} \) and Poisson kernel \( P^{\text{in}}_{\lambda} \) for it, provided \( \lambda \) is not an eigenvalue of the auxiliary inner problem. Then we can represent the resolvent kernel \( G^{\text{in}}_{\lambda} \) in a form similar to one given in Lemma 1,

\[
G^{\text{in}}_{\lambda}(x, y) = G^{\text{in}}_{\lambda-1}(x, y) + \sum_{s=1}^{M} \left( \frac{\lambda + 1}{\lambda_s + 1} \right)^N \sum_{t=1}^{N_s} \varphi_{st}(x) \varphi_{st}(y) \frac{\lambda - \lambda_s}{\lambda - \lambda_s} + \sum_{\lambda_p} \sum_{t=1}^{N_p} \varphi_{tp}(x) \varphi_{tp}(y) \left( \frac{\lambda + 1}{\lambda_p + 1} \right)^N + \sum_{s=1}^{N} (\lambda + 1)^s G^{s+1}_{\lambda-1}(x, y),
\]

\[
P^{\text{in}}_{\lambda}(x, y) = \frac{\partial}{\partial n_y} G^{\text{in}}_{\lambda}(x, y) \bigg|_{y \in \partial(\Omega_1 \cup \Omega_2)}.
\]

Using the representation of the Poisson kernel of the outer problem, described in Lemma 2, we get the following integral equation for the boundary value of the scattered wave of the perturbed operator on the wall \( \gamma_3 \):

\[
g = P^0 \{ P^3 (g + \psi_3) \}_{\Sigma_3} + P^0 P^{\text{in}} g.
\]

(15)
The operator $\mathcal{P}^0\{\mathcal{P}^3 g|_{\Sigma_{33}} + \mathcal{P}^{\text{in}} g|_{\Sigma_{33}}\}$ is represented in form of a sum of the contracting operator

$$K^0 g = \mathcal{P}^0\left\{\mathcal{P}^3 g|_{\Sigma_{33}} + \mathcal{P}^{\text{in}} g|_{\Sigma_{23}}\right\} + o(1) g$$

and the finite dimensional one with the singular coefficient

$$K_\lambda = \mathcal{P}^0 \sum_{s=1}^{M} \left( \frac{\lambda + 1}{\lambda_s + 1} \right)^N \sum_{n_3} \frac{\partial \varphi_{n_3}}{\partial n_x} \frac{\partial \varphi_{n_3}}{\partial n_y} \varepsilon \cos \theta,$$

where $z = \varepsilon \cos \theta$ is a point on the circle $\Sigma_5$, $\theta = (n_{x_0}, x - x_0)$, and the inhomogeneous term is combined of the boundary values of the unperturbed scattered wave.

The proof of the fact, that the operator $K_0$ is a contracting one is based on the maximum principle for the Helmholtz equation. Really, for given function $g$, defined on $\gamma_3$ and $\lambda = -1$ we have due to the maximum principle:

$$|\mathcal{P}^3 g|_{\Sigma_{33}} \leq \frac{1}{2} \sup_{\gamma_3} |g|,$$

$$|\mathcal{P}^{\text{in}} g|_{\Sigma_{23}} \leq \frac{1}{2} \sup_{\gamma_3} |g|,$$

hence once more using maximum principle we have

$$|\mathcal{P}^3 g + \mathcal{P}^{\text{in}} g|_{\Sigma_3} \leq \frac{1}{2} \sup_{\gamma_3} |g|.$$

It means, that the operator $K^0$ is contracting

$$|K^0 g| = \left| \mathcal{P}^0 \left\{\mathcal{P}^3 g|_{\Sigma_{33}} + \mathcal{P}^{\text{in}} g|_{\Sigma_{23}}\right\} \right| \leq \frac{1}{2} \sup_{\gamma_3} |g|.$$  \hspace{1cm} (16)

Now the equation (15) can be rewritten in the form

$$g = (1 - K^0)^{-1} K_\lambda g + (1 - K^0)^{-1} \mathcal{P} \psi|_{\Sigma_{33}},$$  \hspace{1cm} (17)

including a finite dimensional operator $K_\lambda$, $\text{dim} K_\lambda = \sum_{s=1}^{M} N_s$, which depends on $\lambda$ as a rational function. The equation (15) has an unique solution, if

$$\det[1 - (1 - K^0)^{-1} K_\lambda] \neq 0.$$  \hspace{1cm} (18)

The perturbed scattered waves exist for real values of the spectral parameter, $\lambda$ which satisfy the last condition (18).
Vice versa, it is easy to check, that the complex roots of the determinant (18) coincide with resonances, and real roots can be both resonances or eigenvalues. From the Rouchet theorem follows, that the total multiplicity of them is equal to \( \sum_{s=1}^{M} N_s \). The corresponding resonance states \( \varphi_\lambda \) can be produced from eigenstates of the "nonlinear" (in \( \lambda \)) eigenvalue problem

\[
g = (1 - K^0)^{-1} K_\lambda g,
\]

\[
\det [1 - (1 - K^0)^{-1} K_\lambda] = 0
\]

by Poisson integrals,

\[
\varphi_\lambda = \begin{cases} 
  \mathcal{P}^3 g, & x \in \Omega_3, \\
  \mathcal{P}^{\text{in}} g, & x \in (\Omega_1 \cup \Omega_2),
\end{cases}
\]

which give growing functions, called resonance states, for complex roots of the determinant or bounded and/or decreasing ones for real roots. These functions are solutions of the homogeneous problem, when the spectral parameter coincides with the roots.

The existence and smoothness of the Poisson kernel of the nonperturbed outer problem \( \mathcal{P}^3 \) is guaranteed, if no resonances of the problem are situated in a neighbourhood of the group.

II. The shifts and splittings of eigenvalues produced by attaching the channel can be calculated in a similar way. Let us suppose, that the outer domain is bounded, and there exists a separated group of eigenvalues of the perturbed inner problem and a group of eigenvalues of the outer problem, which both are situated in a domain \( G \) on the complex plane of the spectral parameter \( \lambda \) and asymptotically separated from the rest of the spectrum of these problems.

**Theorem 2.** If the group of eigenvalues of the auxiliary inner problem and the group of eigenvalues of the outer problem exist in \( G \) and both of them are asymptotically separated from the rest of the spectrum of both problems, then for the corresponding joint perturbed problem with the narrow channel in the domain \( G \), there exist a group of eigenvalues, whose common multiplicity is equal to the multiplicity of joint initial group.

**Proof.** The integral equation for the boundary values of the resolvent can be represented in the form used in the course of the proof of the preceding theorem, with the only difference, that now we consider the homogeneous equations for the boundary values \( g_3 \) of the solution on the wall \( \gamma_3 \)

\[
g = \mathcal{P}^0 \left\{ \mathcal{P}^{\text{in}} g \left|_{\Sigma_{23}} \right. + \mathcal{P}^3 g \left|_{\Sigma_{33}} \right. + G^3 \left|_{\Sigma_{33}} \right. \right\}
\]

\[
= (\mathcal{P}^{-1} + O(\varepsilon)) \left\{ \mathcal{P}^{\text{in}} \left( \lambda + 1 \right) \sum_{\nu=1}^{N} \frac{\partial \varphi_{\lambda \nu}}{\partial \eta_{\eta \nu}} \frac{\partial \varphi_{\lambda \nu}}{\partial \eta_{\eta \nu}} \cos \theta + O^{\text{in}}(\varepsilon) \right\} g \left|_{\Sigma_{33}} \right. 
\]

\[
+ \left\{ \mathcal{P}^3 \left( \lambda + 1 \right) \sum_{\nu=1}^{N} \frac{\partial \varphi_{\lambda \nu}^3}{\partial \eta_{\eta \nu}} \frac{\partial \varphi_{\lambda \nu}^3}{\partial \eta_{\eta \nu}} \cos \theta + O^3(\varepsilon) \right\} g \left|_{\Sigma_{33}} \right. 
\]

\[
+ \mathcal{P}^0 \left. G^3 \right|_{\Sigma_{33}}.
\]
Combining the contracting part of the corresponding integral operator in form

\[ Kg = [\mathcal{P}_1 + O(\varepsilon)] \left[ \{ \mathcal{P}^{\text{in}}_1 + O^{\text{in}}(\varepsilon) \} \right]_{\Sigma_33} + \{ \mathcal{P}^{\text{in}}_3 + O^3(\varepsilon) \} g \bigg|_{\Sigma_33}, \]

and the finite dimensional part in the form

\[ (\mathcal{P}_1 + O(\varepsilon)) \sum \left( \frac{\lambda + 1}{\lambda_p + 1} \right)^N \sum \frac{\partial \varphi_E}{\partial n_x} \frac{\partial \varphi_E}{\partial n_y} \cos \theta g \bigg|_{\Sigma_33} + (\mathcal{P}_1 + O(\varepsilon)) \sum \left( \frac{\lambda + 1}{\lambda^2 + 1} \right) \frac{\partial \varphi_E}{\partial n_x} \frac{\partial \varphi_E}{\partial n_y} g \bigg|_{\Sigma_33} = K \lambda g, \]

we reduce the problem to the situation, observed in Theorem 1. The further proof follows the same way.

The described method of analysis of the perturbation of resonances and eigenvalues can be applied to the Schrödinger equation in domains, joined by thin channels. Similar problems arise in nanoelectronics, when considering the ballistic movement of electrons in "horizontal" nanoelectronic channels, see [8]. One can show, that the shifts and splittings of resonances and eigenvalues are defined in the main order, by the auxiliary spectral problem, and the removing of the "last wall" produces only exponentially small perturbations. In particular, the shifts of resonances, generated by the eigenvalues of the inner problem from the real axis will be exponentially small in comparing to the shifts of them along the real axis. The analysis of examples will be published later.

**Appendix I. Auxiliary Spectral Problem for the Laplacian in** \( \Omega_1 \cup \Omega_2 \) \n
**with Dirichlet Boundary Condition at** \( (\partial \Omega_1 \cup \partial \Omega_2) \setminus \gamma_1 \).

In this case the system of integral equations containing the contracting operator, is simplified, since \( g_3 \) is absent, so we get only one equation to solve for \( g_1 \). We write it down for the resolvent kernel's boundary values \( g \):

\[ g_\lambda = \mathcal{P}^0 \{ \mathcal{P}_1 g_\lambda + G^1_\lambda \} \bigg|_{\Sigma_{12}} + \mathcal{P}^0 \mathcal{P}_2 g_\lambda = Kg + \mathcal{P}^0 G^1_\lambda \bigg|_{\Sigma_{12}}. \tag{19} \]

If this equation is solved, then the perturbed resolvent can be reconstructed by the solution of the auxiliary Dirichlet problems in \( \Omega_1, \Omega_2 \)

\[ G_{\text{perturbed}}(\lambda) = \begin{cases} 
G_{\text{unperturbed}}(\lambda) + \mathcal{P}_1(\lambda) g_\lambda, & x \in \Omega_1, \\
\mathcal{P}_2(\lambda) g_\lambda, & x \in \Omega_2, 
\end{cases} \]

provided, that \( \lambda \) is not eigenvalue of the perturbed problem.

**Theorem 1.1.** Equation (19) is an equation with a compact operator represented in form of a sum of contracting operator and a finitedimensional operator, depending on the spectral parameter as a rational function, if \( \lambda \) is situated in a neighbourhood of multiple eigenvalue \( \lambda_1 \) of the unperturbed inner problem.
Proof. Using formula (11) for the Poisson kernel in the disk $K_f^1$ and a similar formula for $\Omega_1$, we get the representation for the operator $K$

$$K = \{P_{-1} + O^0(\varepsilon)\} \left\{ P_{-1}^2 + \left( \frac{\lambda + 1}{\lambda_1 + 1} \right)^{N+1} \sum_{s=1}^{m} \varphi_{s1}(x) \frac{\partial \varphi_{s1}(y)}{\partial n_y} \right\}_{\Sigma_{12}} + O^1(\varepsilon) \right\},$$

where $\{\varphi_{s1}\}$ are the eigenvectors of the inner operators and $O^0(\varepsilon), O^1(\varepsilon)$ are integral operators with small norm, $||O^0(\varepsilon)||_{C} \leq \text{const} \varepsilon, ||O^1(\varepsilon)||_{C} \leq \text{const} \varepsilon$. Using the boundedness of the Poisson operators $P_{0-1}, P_{-1}^2$ in $C(\delta) = C(\Sigma_1)$ we can rewrite the operator $K$ in form

$$K = P_{0-1}^0 P_{-1}^2$$

$$+ \left[ P_{0-1}^0 \left( \frac{\lambda + 1}{\lambda_1 + 1} \right)^{N+1} + O^0(\varepsilon) \right] \frac{1}{\lambda_1 - \lambda} \left[ \sum_{s=1}^{m} \varphi_{s1}(x) \frac{\partial \varphi_{s1}(y)}{\partial n_y} \right]_{x \in \Sigma_{12}}$$

$$+ O^0(\varepsilon)P_{-1}^2|_{\Sigma_{12}} + \{P_{-1} + O^0(\varepsilon)\}O^1(\varepsilon).$$

Replacing $\varphi_{s1}(x)|_{x \in \Sigma_1}$ by $\frac{\partial \varphi}{\partial n_x}|_{\gamma_1}$, $\theta = (x - x_0, n_{x_0})$, we get the following form of the equation for $g$:

$$\left\{ I - P_{0-1}^0 P_{-1}^2 + O(\varepsilon) \right\} g =$$

$$\left[ P_{0-1}^0 \left( \frac{\lambda + 1}{\lambda_1 + 1} \right)^{N+1} + O^0(\varepsilon) \right] \cos \theta \frac{1}{\lambda_1 - \lambda} \sum_{s=1}^{m} \frac{\partial \varphi_{s1}(x)}{\partial n_x} \left| \frac{\partial \varphi_{s1}(y)}{\partial n_y} \right|_{\gamma_1} + g + P^0 G|_{\Sigma_3}.$$

The operator on the left part is obviously invertible, hence the whole equation can be reduced to the finite dimensional one

$$g = \left\{ I - P_{0-1}^0 P_{-1}^2 + O(\varepsilon) \right\}^{-1} \left[ P_{0-1}^0 \left( \frac{\lambda + 1}{\lambda_1 + 1} \right)^{N+1} + O^0(\varepsilon) \right] \cos \theta \cdot$$

$$\frac{1}{\lambda_1 - \lambda} \sum_{s=1}^{m} \frac{\partial \varphi_{s1}(x)}{\partial n_x} \left| \frac{\partial \varphi_{s1}(y)}{\partial n_y} \right|_{\gamma_1} g + \left\{ I - P_{0-1}^0 P_{-1}^2 + O(\varepsilon) \right\}^{-1} P^0 G|_{\Sigma_3}.$$

Solution of this equation demands calculation of the corresponding eigenvalues as roots of characteristic determinant

$$\text{det} \left[ I - \left\{ I - P_{0-1}^0 P_{-1}^2 + O(\varepsilon) \right\}^{-1} \left[ P_{0-1}^0 \left( \frac{\lambda + 1}{\lambda_1 + 1} \right)^{N+1} + O^0(\varepsilon) \right] \cos \theta \cdot$$

$$\frac{1}{\lambda_1 - \lambda} \sum_{s=1}^{m} \frac{\partial \varphi_{s1}(x)}{\partial n_x} \left| \frac{\partial \varphi_{s1}(y)}{\partial n_y} \right|_{\gamma_1} \right].$$

The last can be easily calculated by reducing it to the corresponding matrix problem. If the $\lambda$ under consideration is not an eigenvalue of the described finite dimensional problem, we have the existence and uniqueness of the solution $g$.

Conversely, the eigenvalues of the finite-dimensional matrix problem prove to be eigenvalues of the perturbed operator and the corresponding eigenfunctions can be reconstructed from the eigenvectors of the matrix problem.
Appendix II. Auxiliary Spectral Problem in $\Omega_2 \cup \Omega_3$

Let us consider the spectral problem with the homogeneous Dirichlet boundary conditions on $\partial(\Omega_2 \cup \Omega_3) = (\partial \Omega_3 \cup \partial \Omega_2) \setminus \gamma_3$. In this case the system of integral equations (7) is simplified due to the absence of $g_1, g_1 = 0$, so we get only one equation for the scattered wave

$$g = \mathcal{P}_3^0 \{ \mathcal{P}_3^+ g + \psi_\nu \} \big|_{\Sigma_3} + \mathcal{P}_2^0 \mathcal{P}_2 g. \tag{20}$$

Here $\psi_\nu$ is the scattered wave in $\Omega_3$, and $\mathcal{P}_3^{+}\lambda\rangle$ is the boundary value of the Poisson kernel

$$\mathcal{P}_3^{+}\lambda\rangle (x, y) = \frac{\partial G_{\lambda+it}(x, y)}{\partial n_y}, \quad y \in \partial \Omega_3, \ x \in \Omega_3.$$

The scattered wave in $\Omega_2 \cup \Omega_3$ is constructed as a two-component function:

$$\psi_\nu^3 = \psi_\nu + \mathcal{P}_3^+ g,$$

$$\psi_\nu^2 = \mathcal{P}_2 g.$$

**Theorem 1.2.** The integral equation (20) possesses a unique solution in the space of continuous functions $C(\gamma_3)$, provided $\lambda$ is a generic point of continuous spectrum of the outer problem, i.e. the scattered waves depend smoothly on the spectral parameter in a neighbourhood of $\lambda$.

**Proof.** If the spectral parameter $\lambda$ is chosen in such a way that maximum principle is true, e.g. $\lambda = -1$, then the operator

$$\mathcal{P}_3^{+}\lambda\rangle = \mathcal{P}_3^{+}\lambda\rangle + \frac{i}{4} \int_{\Sigma_1} \psi_\nu(x, \lambda) \frac{\partial \psi_\nu(y, \lambda)}{\partial n_y} \, d\nu +$$

$$\frac{\partial}{\partial n_y} \left\{ \sum_{\nu=1}^{N} (\lambda + 1)^{\nu} G_{\nu-1}^{s+1}(x, y) + \nu. p. \int_{\lambda} \int_{\Sigma_1} d\nu \, dz \frac{(\lambda + 1)}{8\pi^2} \left( \frac{\lambda + 1}{z + 1} \right)^{N+1} \frac{\psi_\nu(x, z) \psi_\nu(y, z)}{z - \lambda} \right\}$$

and the similar formula for $\mathcal{P}_3^{\lambda\rangle}$ inside the disk $K_3$

$$\mathcal{P}_3^{\lambda\rangle} = \frac{\partial G_0}{\partial n_y} = \frac{\partial G_{-1}}{\partial n_y} + \sum_{\nu=1}^{N} (\lambda + 1)^{\nu} \frac{\partial G_{-1}^{s+1}(x, y)}{\partial n_y}$$

$$+ \sum_{\nu=1}^{N} \left( \frac{\lambda + 1}{\lambda_\nu + 1} \right)^{N+1} \frac{1}{\lambda_\nu - \lambda} \frac{\partial \varphi_{ps}(y)}{\partial n_y} \varphi_{ps}(x).$$
All the eigenvalues \( \lambda_p \) of Laplace operator in the disk \( K_3 = K_{35}^3 \) with zero boundary conditions are much greater, than \( \lambda \), if \( \varepsilon \) is small enough. Hence using the smoothness of iterated resolvent kernels, we can conclude, that in both cases the main part of \( \mathcal{P}^+_{3}(\lambda) \), \( \mathcal{P}^0_{\lambda} \) is given by \( \mathcal{P}^+_{3(-1)} \), \( \mathcal{P}^0_{(-1)} \) respectively, the rest being represented by operators with smooth kernels on a small domain, since the singularities do not destroy the convergence of the integrals:

\[
G_{-1} \frac{\partial}{\partial n} G_{-1} \sim \int \ln |r| \frac{1}{|r|} d^2x < \text{const},
\]

and hence a small norm \( \sim \varepsilon \). The same arguments could be applied to \( \Omega_2, \mathcal{P}_{2(\lambda)} \).

Taking into account, that all Poisson operators are bounded in the space of continuous functions, we get

\[
g = \{ \mathcal{P}^0_{-1} + \mathcal{O}(\varepsilon) \} \{ \mathcal{P}^+_{3(-1)} g + \mathcal{O}_3(\varepsilon) g + \psi_\nu \} \bigg|_{\Sigma_{33}} + \{ \mathcal{P}^0_{-1} + \mathcal{O}(\varepsilon) \} \{ \mathcal{P}^+_{2(-1)} g + \mathcal{O}_2(\varepsilon) g \} \bigg|_{\Sigma_{33}} + \mathcal{P}^0_{-1} \mathcal{P}^+_{2(-1)} g + \mathcal{O}(\varepsilon) g.
\]

Here \( \mathcal{O}(\varepsilon) \) is an operator with the small norm in \( C_{\gamma_3}, |\mathcal{O}(\varepsilon)|_C \leq \text{const} \varepsilon \). Since the sum of a contracting operator and a small operator is a contracting operator as well, we see, that for small \( \varepsilon \) the equation (20) has a unique solution \( g \), which possesses the same generic properties, as the scattered wave \( \psi_\nu \).

Having the representation for \( g \) via the contracting operator \( K \):

\[
g = \left[ I - \mathcal{P}^0 \left\{ \mathcal{P}^+_{3} \big| \Sigma_{33} + \mathcal{P}^{2} \big| \Sigma_{32} \right\} \right]^{-1} \mathcal{P}^0 \psi_\nu \bigg|_{\Sigma_{3}} = [I - K]^{-1} \mathcal{P}_3^0 \psi_\nu \bigg|_{\Sigma_{3}}, \quad (21)
\]

we can reconstruct the inner and outer components of the perturbed scattered waves, just by solving the corresponding Dirichlet problems in \( \Omega_2, \Omega_3 \). Note, that the Poisson kernels for \( \lambda \neq 0 \) are decreasing at infinity generally not slower than the main singular solution.

One can calculate the scattering amplitude from \( \psi_3^0 \). In fact, using the asymptotics for the resolvent kernel for \( x \to \omega \infty \)

\[
G_{3,\lambda}^+(x,y) \equiv \psi_\omega(x,\lambda)G_{3,\lambda}^+(0,y),
\]

and the symmetry of Green functions we have

\[
\mathcal{P}^+_{3,\lambda}(x,y) \equiv \frac{\partial \psi_\omega}{\partial n_x} G_{3,\lambda}^+(0,y).
\]

Then the formula for the amplitude follows immediately from the representation for the perturbed scattered wave:

\[
f_{\text{pert}}(\nu,\omega) = f_{\text{unperturbed}}(\nu,\omega) + \int_{\gamma_3} g_\nu(x) \frac{\partial \psi_\omega}{\partial n_x} dx.
\]
Note, that the perturbed amplitude is symmetrical in respect to the unit vectors \( \omega, \nu \). Really, we have the following asymptotics

\[
\psi_{\nu}|_{\Sigma_3} \sim \frac{\partial \psi_{\nu}}{\partial n_y} \sqrt{\varepsilon^2 - |y - x_0|^2},
\]

\[
g_{\nu}(x) = [I - K]^{-1} \mathcal{P}^0 \sqrt{\varepsilon^2 - |y - x_0|^2} \frac{\partial \psi_{\nu}}{\partial n_y}.
\]

Here \([I - K]^{-1} = I + K + K^2 + \ldots\) is the integral operator (21), and we can assume, that \( \frac{\partial \psi_{\nu}}{\partial n_y} \sim \frac{\partial \psi_{\nu}}{\partial n_{x_0}} \) in the generic case. Hence for small \( \varepsilon \) we get

\[
f_{\text{pert}}(\nu, \omega) = f_{\text{unpert}}(\nu, \omega) + \frac{\partial \psi_{\nu}}{\partial n_x} \frac{\partial \psi_{\omega}}{\partial n_x}(x_0) \delta f,
\]

\[
\delta f = \int_{\gamma_3} [I - K]^{-1} \mathcal{P}^0 \sqrt{\varepsilon^2 - |y - x_0|^2} dx_3 = \int_{\gamma_3} \mathcal{P}^0 \sqrt{\varepsilon^2 - |y - x_0|^2} dx_3 + \ldots.
\]

Note, that the first member in the right side is the solution of the Dirichlet problem with the given boundary values on the circle \( \Sigma_3 \)

\[
\begin{cases}
\sqrt{\varepsilon^2 - |y - x_0|^2} = \varepsilon \cos(n_{x_0}, \theta), & \theta \leq \frac{\pi}{2} \\
0, & \theta > \frac{\pi}{2}
\end{cases}
\]

Being reduced onto the diameter \( \gamma_3 \) of the disk \( K^x_3 \) it gives a standard function

\[
\mathcal{P}^0 \sqrt{\varepsilon^2 - |y - x_0|^2} = \frac{\varepsilon}{\pi} \left[ 1 + \frac{y}{2} + \sum_{m=2}^{\infty} \frac{m^2 + 1}{(m^2 - 1)^2} y^m \right],
\]

which arises as a necessary detail of the suggested construction.

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