ON A DIRECT AND INVERSE SCATTERING PROBLEM FOR A BOUNDARY VALUE PROBLEM WITH DISCONTINUOUS COEFFICIENT

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Abstract. The author studies the inverse scattering problem for a boundary value problem of a generalized one dimensional Schrödinger's type with a discontinuous coefficient. The solutions of the considered eigenvalue equation is presented and its scattering function that satisfies some properties is induced. The discrete spectrum is studied and the resolvent of the considered problem is given. The expansion by eigenfunctions of the given problem is obtained and hence Parseval's equality is formulated. The scattering data are determined and hence the inverse scattering problem is formulated and completely solved.

Introduction

Consider the boundary value problem corresponding to the generalized form of the one-dimensional Schrödinger equation

\[-y'' + q(x)y = \lambda \rho(x)y, \quad x \in [0, \infty) \tag{1}\]

and the boundary condition

\[y'(0) - h y(0) = 0, \tag{2}\]

where \(\lambda\) is a complex number and \(h\) is a real number. We assume that the function \(q\) is real and satisfies the condition

\[\int_0^\infty x |q(x)| dx < \infty \tag{3}\]

which is assumed to hold throughout the paper. The function \(\rho\) is defined as

\[\rho(x) = \begin{cases} \alpha^2, & 0 \leq x \leq a \\ 1, & a < x < \infty, \quad \alpha \neq 1, \quad \alpha \in R^+ \end{cases} \tag{4}\]

It is worth noting that a differential equation is given in the direct problems and a particular solution is sought from certain functions. In the inverse problem a solution is given and a particular differential equation is sought from a given class of equations.

The modern trend in quantum scattering theorems is to solve the inverse problem by the so-called scattering data. These scattering data are defined as the collection of quantities \(\{S(s); \lambda_1, \lambda_2, \ldots, \lambda_m; M_1, M_2, \ldots, M_m\}\). Here \(S(s)\) is the scattering function of the problem and \(\lambda_1, \lambda_2, \ldots, \lambda_m\) are the eigenvalues and \(M_1, M_2, \ldots, M_m\) are the normalization coefficients.

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In this paper, we can state the inverse scattering problem for the problem (1) — (2) to indicate a method of determining the potential function \( q \) and the density (weight) function \( \rho \) and \( h \). In the present paper this problem will be completely solved. It should be mentioned that equation (1) represents the equation of motion of inhomogenous vibrating string which is in the form [14, 15]

\[
\frac{\partial}{\partial x} \left[ K(x) \frac{\partial u}{\partial x} \right] - q(x) u = \rho(x) \frac{\partial^2 u}{\partial t^2}.
\]

The Robin condition (2) means that the string is elastically constrained. Moreover, if the potential function \( q \) in equation (1) satisfies condition (3) then equation (1) reduces to a simpler equation representing intrinsic vibration of a loaded string [14] or the equation of infinite inhomogenous semi-string with density \( \rho = \rho(x) \) [13]. It may be pointed out that the direct and inverse problem is considered previously in different cases. In case \( \rho \equiv 1 \) the direct and inverse problem of (1) with \( y'(0) - hy(0) = 0 \) [2, 3] has been solved earlier by the so called spectral distribution function while the problem (1) with \( y(0) = 0 \) has been studied in the works [3, 16] by the inverse scattering method. Furthermore, the inverse scattering problem of one-dimensional Schrodinger's eigenvalue problem with a discontinuous coefficient was studied when \( y(0) = 0 \) and \( y'(0) = 0 \) [6, 11, 12]. In this paper I have extended the previous results by considering (1) with \( y'(0) - hy(0) = 0 \) and solve the inverse problem by the inverse scattering method. The organization of this paper is as follows:

We collect in Section 1 certain solutions of equation (1) from [1, 2] which we shall use subsequently. Moreover, we define the so called scattering function and we give its asymptotic behaviour. In Section 2 we shall study the discrete spectrum of (1) — (2) and we obtain the resolvent of (1) — (2) and hence we formulate Parseval's equality. As will be seen in Section 3, the collection of quantities \( \{ S(s); -\tau_1^2, M_n, n = 1, m \} \) is called the scattering data of (1) — (2), where \( S(s) \) is the scattering function and the numbers \( -\tau_1^2, -\tau_2^2, \ldots, -\tau_m^2 \) are eigenvalues. The numbers \( M_1, M_2, \ldots, M_m \) are called the normalization coefficients of (1) — (2). In addition, for the normalized eigenfunctions, the following asymptotic formulas hold as \( x \to \infty \)

\[
U(x, s) = \exp(-isx) - S(s) \exp(isx) + O(1)
\]

and

\[
U(x, i\tau_n) = \exp(-\tau_n x) [M_n + O(1)], \quad n = 1, m.
\]

This provides a complete description of the behaviour at infinity of all radial wave functions \( U(x, s) \). Finally, Section 4 will be aimed to solve the inverse scattering problem of (1) — (2). The inverse scattering problem is being extended to a more general case when \( \rho(x) \neq 1 \). In terms of the scattering data, the potential function \( q(x) \) and weight function \( \rho(x) \) and \( h \) are defined uniquely. This is a straightforward consequence of Theorem 8.
1. Solutions for Equation (1) and its Scattering Function

Denote by \( \varphi(x, \lambda) \) and \( \psi(x, \lambda) \) the solutions of (1) on the interval \([0, a]\), satisfying the initial conditions

\[
\begin{align*}
\varphi(0, \lambda) &= 1, & \varphi'(0, \lambda) &= h, \\
\psi(0, \lambda) &= 0, & \psi'(0, \lambda) &= 1.
\end{align*}
\]

Lemma 1. Let \( \lambda = s^2 \). Then

\[
\begin{align*}
\varphi(x, s) &= \cos s\alpha x + \frac{h}{s\alpha} \sin s\alpha x + \frac{1}{s\alpha} \sin s\alpha x \\
&\quad + \frac{1}{s\alpha} \int_0^x \sin s\alpha(x - t)q(t)\varphi(t, s)dt
\end{align*}
\]

and

\[
\begin{align*}
\psi(x, s) &= \frac{\sin s\alpha x}{s\alpha} + \frac{1}{s\alpha} \int_0^x \sin s\alpha(x - t)q(t)\psi(t, s)dt.
\end{align*}
\]

For the proof see [1].

Lemma 2. Let \( s = \sigma + i\tau \) with \( 0 \leq \arg s < \pi \). Then there exists \( s_0 > 0 \) such that, for \( |s| > s_0 \), the estimates

\[
\varphi(x, s) = O\{\exp(|t|x\alpha)\}
\]

and

\[
\psi(x, s) = O\left\{\frac{1}{s} \exp(|t|x\alpha)\right\}
\]

are valid; more precisely

\[
\begin{align*}
\varphi(x, s) &= \cos s\alpha x + O\left\{\frac{1}{s} \exp(|t|x\alpha)\right\} \\
\psi(x, s) &= \frac{\sin s\alpha x}{s\alpha} + O\left\{\frac{1}{s^2} \exp(|t|x\alpha)\right\}.
\end{align*}
\]

These estimates are uniform with respect to \( x \in [0, a] \). See [1].

On the interval \((a, \infty)\) equation (1) becomes \(-y'' + q(x)y = s^2y\). Then, [2], for any \( s \) from the closed upper half plane, the above equation has a solution \( F(x, s) \) on the form

\[
F(x, s) = \exp(isx) + \int_x^\infty K(x, t) \exp(ist)dt; \quad a < x < \infty
\]

where the kernel \( K(x, t) \) satisfies the inequality

\[
|K(x, t)| \leq \frac{1}{2} \sigma \left(\frac{x + \tau}{2}\right) \exp\left\{\sigma_1(x) - \sigma_1\left(\frac{x + \tau}{2}\right)\right\}
\]

and the condition

\[
\frac{dK(x, x)}{dx} = -\frac{1}{2} q(x).
\]

Here

\[
\sigma(x) = \int_x^\infty |q(t)| \quad \text{and} \quad \sigma_1(x) = \int_x^\infty \sigma(t)dt.
\]
The solution $F(x, s)$ is an analytic function of $s$ in the upper half plane $\tau \geq 0$ and is continuous on the real line. The following estimates hold

$$|F(x, s)| \leq \exp\{-\text{Im} sx + \sigma_1(x)\}$$

and

$$|F'(x, s) - is\exp(isx)| \leq \sigma(x)\exp\{-\text{Im} sx + \sigma_1(x)\}.$$  

**Theorem 1.** The identity

$$\frac{2is\varphi(x, s)}{W(s)} = F(x, -s) - S(s)F(x, s)$$

holds for all real values $s \neq 0$, where

$$W(s) = F'(0, s) - hF(0, s) \quad \text{and} \quad S(s) = \frac{W(-s)}{W(s)} = \frac{S(s)}{W(s)} = [S(-s)]^{-1}.$$  

**Proof.** Since the functions $F(x, s)$ and $F(x, -s)$, form a fundamental system of solutions to equation (1) for all real $s \neq 0$, thus

$$\varphi(x, s) = C_1(s)F(x, s) + C_2(s)F(x, -s).$$

Condition (5) yields

$$C_1(s) = \frac{-W(-s)}{2is} \quad \text{and} \quad C_2(s) = \frac{W(s)}{2is}, \quad \text{where} \quad W(s) = F'(0, s) - hF(0, s).$$

Hence, $\varphi(x, s) = (2is)^{-1}[W(s)F(x, -s) - W(-s)F(x, s)]$. Since $q(x)$ is real, thus $W(-s) = \overline{W(s)}$ and hence $W(s) \neq 0$ for all real $s \neq 0$. Therefore

$$\frac{2is\varphi(x, s)}{W(s)} = F(x, -s) - S(s)F(x, s),$$

with

$$S(s) = \frac{W(-s)}{W(s)} = \left[ \frac{W(s)}{W(-s)} \right]^{-1} = \left[ \frac{W(s)}{W(-s)} \right]^{-1}$$

as claimed.

**Definition 1.** The function $S(s)$ is called the scattering function of equation (1) with initial conditions (5).
Theorem 2. For large real \( s \neq 0 \), \( |s| \to \infty \) the following asymptotics formula
\[
S(s) - S_0(s) = O \left( \frac{1}{s^2} \right)
\] holds, where
\[
S_0(s) = \exp(-2isa) \left[ (s\alpha^2 - ih) \frac{\sin s\alpha}{\alpha} - (is + h) \cos s\alpha \right] \cdot 
\left[ (s\alpha^2 + ih) \frac{\sin s\alpha}{\alpha} + (is - h) \cos s\alpha \right]^{-1}.
\]

Proof. It is clear that \( \varphi(x, s) \) and \( \psi(x, s) \) form a fundamental system of solutions of equation (1) on the interval \( [0, a] \), then
\[
F(x, s) = d_2(s)\varphi(x, s) + d_2(s)\psi(x, s).
\]
Now, condition (5) yield
\[
d_1(s) = F(0, s) \text{ and } d_2(s) = W(s).
\]
Thus, we have
\[
F(x, s) = F(0, s)\varphi(x, s) + W(s)\psi(x, s).
\]
Formula (14) gives
\[
W(s) = F'(a, s)\varphi(a, s) - F(a, s)\varphi(a, s).
\]
Taking into account (6) and (8) to obtain
\[
W(s) = \exp(isa) \left[ (s\alpha^2 + ih) \frac{\sin s\alpha}{\alpha} + (is - h) \cos s\alpha \right] + O \left( \frac{1}{s^2} \right).
\]
Thus we conclude that
\[
S(s) = W(-s)[W(s)]^{-1}
= \exp(-2isa) \left[ (s\alpha^2 - ih) \frac{\sin s\alpha}{\alpha} - (h + is) \cos s\alpha \right] \cdot 
\left[ (s\alpha^2 + ih) \frac{\sin s\alpha}{\alpha} + (is - h) \cos s\alpha \right]^{-1} + O \left( \frac{1}{s^2} \right)
= S_0(s) + O \left( \frac{1}{s^2} \right),
\]
where \( S_0(s) \) is defined by (13).
2. The Discrete Spectrum and the Resolvent of (1) – (2)

This section is devoted to study the discrete spectrum and to obtain the resolvent of (1) – (2). Making use of [3] to prove the following lemma.

**Lemma 3.** The necessary and sufficient conditions that \( \lambda \neq 0 \) be an eigenvalue of (1) – (2) are that

\[
\lambda = s^2; \quad \tau > 0 \quad \text{and} \quad W(s) = F'(0, s) - hF(0, s) = 0.
\]

**Theorem 3.** The problem (1) – (2) does not have an eigenvalue on positive semi-axis. The set of eigenvalues is no more than countable, and its limit point can lie on the half-axis \( \lambda \geq 0 \). The eigenvalues are lie on the imaginary axis of s-plane and are all simple. They are bounded.

**Proof.** In the sequel, we show that the function \( W(s) \neq 0 \) for real values \( \lambda = s^2 \neq 0 \). Suppose the contrary. Let \( s_0 \in [0, \infty) \), \( s_0 \neq 0 \) such that \( W(s_0) = 0 \). Thus, we have \( F'(0, -s_0) = hF(0, -s_0) \). Since

\[
2i s_0 = W[F(0, -s_0), F(0, s_0)] = F(0, -s_0)F'(0, s_0) - F(0, s_0)F'(0, -s_0)
\]

the assumption leads to a contradiction \( s_0 \neq 0 \).

Therefore, we conclude that (1) – (2) does not have positive eigenvalue and this problem has not a singular spectrum. Since, the function \( W(s) \) is an analytical function in the upper half plane \( \tau > 0 \), its zeros form an at most countable set having 0 as the only possible limit point. Here, we show that the eigenvalues are lie on the imaginary axis of s-plane. Let \( s(s = 0 \text{ or } \text{Im} s > 0) \) be one of the zeros of \( W(s) \). Since

\[
W[\varphi(x, s), F(x, s)] = W(s) = 0,
\]

then

\[
F(x, s) = c\varphi(x, s) \quad \text{and} \quad \lim_{x \to 0} F(x, s) = c
\]

and

\[
F(x, s) = F(0, s)\varphi(x, s). \quad (15)
\]

Formula (15) leads to

\[
\lim_{x \to 0} W \left[ F(x, s_1), F(x, s_2) \right] = 0 \quad (16)
\]

for two arbitrary zeros \( s_1 \) and \( s_2 \) of \( W(s) \). Since \( q(x) \) is real function, we have

\[
W \left[ F(x, s_1), F(x, s_2) \right]_0^\infty = (s_1^2 - s_2^{-2}) \int_0^\infty F(x, s_1)F(x, s_2)dx.
\]

Using (15) and the estimate (9) and (10) to obtain

\[
(s_1^2 - s_2^{-2}) \int_0^\infty F(x, s_1)F(x, s_2)dx = 0.
\]
Taking, in particular, \( s_1 = s_2 \) to have \( s_1 + s_2 = 0 \), i.e. \( s_1 = i \tau_1 \). Hence, the zeros of the function \( W(s) \) can lie only on the imaginary axis. Also, using [2] and the estimate (9) and (16) to show that the zeros of \( W(s) \) are all simple. Finally, since

\[
W(s) = \exp(is\alpha) \left[ (s\alpha^2 + \frac{ih}{\alpha}) \frac{\sin s\alpha}{s} + (is - h) \cos s\alpha \right] + O\left(\frac{1}{s^2}\right) \neq 0
\]

for sufficiently large \( s \), the number \( \lambda = s^2 \) cannot be an eigenvalue of (1) – (2) and we conclude that these eigenvalues are a bounded set. The theorem is completely proved.

**Lemma 4.** All numbers \( \lambda = s^2 ; \tau > 0 \) and \( W(s) \neq 0 \) belong to the resolvent set of the problem (1) – (2). If \( W(s) \neq 0 \) then the resolvent for \( y'' + q(x)y - \lambda \rho y = \rho f; \ x \in [0, \infty), \ f(x) \in L_2(0, \infty; \rho(x)) \) is an integral operator

\[
G_s(\rho f) = \int_0^\infty G(x,t;s)\rho(t)f(t)dt,
\]

where \( G(x,t;s) \) is defined by

\[
G(x,t;s) = -\frac{1}{W(s)} \left\{ \begin{array}{ll}
F(x,s)\varphi(t,s), & t \leq x \\
F(t,s)\varphi(x,s), & t \geq x
\end{array} \right.
\]  

(17)

called the kernel resolvent for the problem (1) – (2).

This lemma can be proved by using variation of parameter method and taking into account condition (2). It follows from variation of parameters that the general solution of the nonhomogenous equation \( y'' + q(x)y - \lambda \rho y = \rho f; \ x \in [0, \infty) \) is

\[
y(x,s) = c_1 F(x,s) + c_2 \varphi(x,s) + \int_0^\infty G(x,t;s)\rho(f)f(t)dt,
\]

where \( G(x,t;s) \) is defined by (17). Since, for values of \( s \) in the upper half-plane, the functions \( F(x,s) \in L_2(0, \infty; \rho(x)) \) and \( \varphi(x,s) \notin L_2(0, \infty, \rho(x)) \) are the solutions of the homogenous equation (1). Therefore \( c_1 F(x,s) + c_2 \varphi(x,s) \) is a solution of (1) that belongs to \( L_2(0, \infty; \rho(x)) \) only for those values \( s \) in the upper half plane when \( c_1 = c_2 = 0 \). So \( y(x,s) \) is the unique solution that belongs to \( L_2(0, \infty; \rho(x)) \) of equation (1). It can be directly checked that the function \( y(x,s) \) satisfies condition (2).

### 3. Eigenfunction Expansion and Parseval’s Equality

In this respect, we obtain the expansion by eigenfunctions of (1) – (2) by Titchmarsh’s Method [4] and the work [5]. Moreover, we formulate Parseval’s equality.

**Lemma 5.** Assume that the function \( f(t) \) is finite and has a continuous derivative in \( L_2(0, \infty; \rho(x)) \) and satisfies the boundary condition (2). Then

\[
\int_0^\infty G(x,t;s)\rho(t)f(t)dt = -\frac{f(x)}{s^2} + \frac{1}{s^2} \int_0^\infty G(x,t;s)h(t)dt,
\]

where \( h(t) = -f''(t) + q(t)f(t) \). Furthermore, if \( \tau > 0 \) and \( |s| \to \infty \), then

\[
\int_0^\infty G(x,t;s)\rho(t)f(t)dt = -\frac{f(x)}{s^2} + O\left(\frac{1}{s^2}\right).
\]  

(18)
Proof. Since \( \varphi(x, s) \) and \( F(x, s) \) satisfy (1) then using (17)
\[
\int_0^\infty G(x,t; s)\rho(t)f(t)dt = -\frac{1}{s^2W(s)} \left[ F(x, s) \int_0^x [\varphi''(x, s) - q(t)\varphi(t, s)]f(t)dt + \right. \\
\left. \varphi(x, s) \int_x^\infty [F''(t, s) - q(t)F(t, s)]f(t)dt \right].
\]
Integrating this identity by parts to arrive at the first part of the lemma. From (6), (9) and (17) it yields that \( G(x,t; s)h(t) = o(1) \) as \( \tau \to 0 \) and \( |s| \to \infty \). Hence (18) follows directly.

The following lemma is well known [5].

Lemma 6. \( \bar{G}(x,t; s) = G(x,t; \bar{s}). \)

In view of these lemmas we prove that the following expansion theorem.

Theorem 4. If the function \( f(x) \) satisfies the conditions of Lemma 5, then its expansion by eigenfunctions of (1) – (2) can be written on form
\[
f(x) = \frac{1}{2\pi} \int_0^\infty \int_0^\infty U(x,s)U(t,-s)\rho(t)f(t)dtds \\
+ \sum_{n=1}^m \int_0^\infty U(x,i\tau_n)U(t,i\tau)\rho(t)f(t)dt,
\]
where \( U(x,i\tau_n) = M_n F(x,i\tau_n) \) and
\[
M_n^2 = \frac{2i\tau_n}{W(i\tau_n)F(0,i\tau_n)}.
\]

Proof. Suppose that \( f(x) \) satisfies the conditions of Lemma 5, then (18) holds. Multiplying both sides of (18) by \( \frac{k}{\pi i} \) and integrating over the semi-circle \( |s| = r \) with respect to \( s \) in the upper half plane of \( s \). Evidently, the integral
\[
\int_0^\infty G(x,t; s)\rho(t)f(t)dt
\]
is a holomorphic function except at the zeroes \( \{i\tau_1, \ldots, i\tau_m\} \) of the function \( W(s) \). Then using [5] to obtain
\[
f(x) = \frac{1}{\pi i} \int_0^\infty s \int_0^\infty \left[ G(x,t; s + io) - G(x,t; s - io) \right]\rho(t)f(t)dtds \\
- \sum_{n=1}^m \Re s \left[ 2s \int_0^\infty G(x,t; s)\rho(t)f(t)dt \right]_{s=i\tau_n} \quad (20)
\]
Let us compute the first term in the right hand side of (20). In order to compute this term, determine \( G(x,t; s + io) \) and then use Lemma 6 to have \( G(x,t; s - io) \). Substituting from (14) into (17) to have
\[
G(x,t; s) = -\frac{1}{W(s)}F(0,s)\varphi(x,s)\varphi(t,s) - R(x,t; s), \quad (21)
\]
where
\[
R(x,t; s) = \begin{cases} \psi(x,s)\varphi(t,s), & t \leq x \\ \varphi(x,s)\psi(t,s), & t \geq x. \end{cases}
\]
Thus, in view of (21) we get
\[ G(x, t; s + io) - G(x, t; s - io) = \frac{\varphi(x, s)2is\varphi(t, s)}{W(s)W(-s)}. \] (22)

Taking into account formula (11) then (22) can be rewritten on the form
\[ G(x, t; s + io) - G(x, t; s - io) = \frac{U(x, s)U(t, s)}{-2is}, \]
where
\[ U(x, s) = \frac{2is\varphi(x, s)}{W(s)} = F(x, -s) - S(s)F(x, s). \]

Therefore
\[ \frac{1}{\pi i} \int_0^\infty \int_0^\infty S[G(x, t; s + io) - G(x, t; s - io)]\rho(t)f(t)dt \, ds = \frac{1}{2\pi} \int_0^\infty \int_0^\infty U(x, s)U(t, -s)\rho(t)f(t)dt \, ds. \] (23)

Here, we compute the second term in the right hand side of (20). Since
\[ \int_0^\infty G(x, t; s)\rho(t)f(t)dt = -\frac{F(0, s)\varphi(x, s)}{W(s)} \int_0^\infty \varphi(t, s)\rho(t)f(t)dt - \psi(x, s) \int_0^x \varphi(t, s)\rho(t)f(t)dt - \varphi(x, s) \int_x^\infty \psi(t, s)\rho(t)f(t)dt. \]

Evidently, the functions \( \varphi(x, s) \) and \( \psi(x, s) \) are analytic and hence
\[ \sum_{n=1}^m \text{Res} \left[ -\int_0^x \psi(x, s)\varphi(t, s)\rho(t)f(t)dt - \int_x^\infty \varphi(x, s)\psi(t, s)\rho(t)f(t)dt \right]_{s=i\tau_n} = 0. \]

Therefore, we have
\[ I = -\sum_{n=1}^m \text{Res} \left[ 2s \int_0^\infty G(x, t; s)\rho(t)f(t)dt \right]_{s=i\tau_n} = -\sum_{n=1}^m \text{Res} \left[ \frac{2s}{W(s)}F(0, s)\varphi(x, s) \int_0^\infty \varphi(t, s)\rho(t)f(t)dt \right]_{s=i\tau_n} = -\sum_{n=1}^m \frac{2i\tau_n}{W(i\tau_n)}F(0, i\tau_n)\varphi(x, i\tau_n) \int_0^\infty \varphi(t, i\tau_n)\rho(t)f(t)dt. \]
Setting 

\[ M_n^2 = \frac{2i\tau_n}{W(i\tau_n)F(0,i\tau_n)} \]

and using (15) it yields that

\[ I = -\sum_{n=1}^{m} \int_0^\infty F(x,i\tau_n)F(t,i\tau_n)\rho(t)f(t)dt \]

\[ = -\sum_{n=1}^{m} \int_0^\infty U(x,i\tau_n)U(t,i\tau_n)\rho(t)f(t)dt, \tag{24} \]

where

\[ U(x,i\tau_n) = M_nF(x,i\tau_n) \]

is the normalization eigenfunctions and the numbers \( M_n, n = 1, m \) are called normalization coefficients. Hence, we obtain from (20), (23) and (24) the following expansion of \( f(x) \) by eigenfunctions of (1) – (2)

\[ f(x) = \frac{1}{2\pi} \int_0^\infty \int_0^\infty U(x,s)U(t,-s)\rho(t)f(t)dtds \]

\[ + \sum_{n=1}^{m} \int_0^\infty U(x,i\tau_n)U(t,i\tau_n)\rho(t)f(t)dt. \]

This is the required result and thus the theorem is proved.

**Definition 2.** The collection of quantities \( \{S(s); -\tau_n^2, M_n, n = 1, m\} \) is called the scattering data of the problem (1) – (2).

It follows immediately from Theorem 4 that the following statement is true.

**Theorem 5.** The following Parseval’s equality holds

\[ \frac{1}{2\pi} \int_0^\infty U(x,s)U(t,-s)ds + \sum_{n=1}^{m} U(x,i\tau_n)U(t,i\tau_n) = \delta(x - 1)\rho^{-1}(x). \tag{25} \]

4. Formulation of Inverse Scattering Problem of (1) – (2)

Here, the inverse scattering problem of the problem (1) – (2) can be stated as follows:

Knowing the scattering data of the problem (1) – (2), can we reconstruct equation (1) and (2), that is, can we find the potential function \( q \) and the density function \( \rho \) and \( h \)?

In this section we give an answer to this question for solving the inverse problem (1) – (2). For this purpose, since the scattering data of (1) – (2) are known, then we can construct the fundamental equation for unique kernel \( K(x,t) \) of formula (7). Using the methods of the work [6] to prove the following theorem:
Theorem 6. The kernel of formula (7) satisfies the fundamental equation

\[ F(x + t) + K(x, t) + \int_{0}^{\infty} K(x, \zeta) F(\zeta + t) d\zeta = 0, \quad a < x < \zeta < \infty, \]  

(26)

where

\[ F(x) = \frac{1}{2\pi} \int_{0}^{\infty} [S_0(s) - S(s)] \exp(isx) ds + \sum_{n=1}^{m} M_n^2 \exp(-\tau_n x) \]  

(27)

and \( S_0(s) \) is defined by (13).

It should be mentioned that for construction the fundamental equation, it is sufficient to know the function \( F(x) \) in its turn, to find \( F(x) \) it is sufficient to know the scattering data. Equation (26) plays an important role in the solution of the inverse scattering problem of (1) – (2) on \((a, \infty)\). If (26) has a unique solution \( K(x, t) \) then the potential \( q(x) \) can be found from (8). As we have just mentioned previously, we prove that the fundamental equation (26) has a unique solution \( K(x, t) \) as \( a < x < \infty \).

Theorem 7. For every fixed \( a < x < \infty \) the fundamental equation (26) has a unique solution in \( L^2(x, \infty) \).

To prove this theorem, it is enough to show that the homogenous equation

\[ f(t) + \int_{x}^{\infty} f(\zeta) F(\zeta + t) d\zeta = 0 \]

has only the zero solution in \( L^2(x, \infty) \). See [6].

Later, in the preceding sections, we have defined the scattering data of (1) – (2) on the form \( \{S(s); -\tau_n^2; M_n, n = \overline{1, m}\} \). Moreover, the asymptotic formulas of the eigenfunctions and the normalized eigenfunctions take the forms

\[ U(x, s) = \exp(-isx) - S(s) \exp(isx) + O(1), \]

and

\[ U(x, i\tau_n) = \exp(-\tau_n x)[M_n + O(1)], \quad n = \overline{1, m}. \]

Furthermore, in this section we have constructed the fundamental equation for the kernel \( K(x, t) \) of (7) and show that it has a unique solution at once. Finally, we give the uniqueness theorem in what follows.

Theorem 8. Assume that the condition (3) and the formula (4) hold. If the scattering data \( \{S(s); -\tau_n^2; M_n, n = \overline{1, m}\} \) are known then the functions \( q \) and \( \rho \) are defined uniquely.
Proof. Consider $S_0(s,a,\alpha,h)$ as the scattering function of the problem $-y'' = s^2\alpha^2 y$, $y'(0) - hy(0) = 0$ then $S_0(s,a,\alpha,h)$ can be defined by formula (13). Taking $a \neq a_1$; $\alpha \neq \alpha_1$ and $h \neq h_1$, thus it is easily seen that $\lim_{s \to \infty} S_0(s,a,\alpha,h)S_0^{-1}(s,a_1,\alpha_1,h_1)$ does not exist and therefore by $S_0(s,a,\alpha,h)$ the numbers $a, \alpha$ and $h$ are defined uniquely. Upon using Theorem 2 it yields that $\lim_{s \to \infty} S_0(s,a,\alpha,h) = 1$, where $S(s)$ is the scattering function of (1) – (2).

Hence, by using $S(s)$ the numbers $a, \alpha$ and $h$ are defined uniquely and then the density function $\rho$ is reconstructed uniquely. Now, we have already obtained fundamental equation (26) in Theorem 6 and proved that this equation has a unique solution $K(x,t)$ such that

$$q(x) = -2\frac{d}{dx}K(x,x)$$

holds as $a < x < \infty$. Thus the potential function $q$ is defined uniquely as $a < x < \infty$ and hence equation (1) can be reconstructed on this interval. Since the functions $F(a,s)$ and $F'(a,s)$ are already defined thus we have the collection of quantities $\{S(s); -r^2_m, M_n; F(a,s); F'(a,s), n = 1, m\}$. The problem now is using these data to define $q$ when $0 \leq x < a$. For this purpose, we construct Weyl’s function $[3, 7]$ for equation (1) on $[0, a]$ by two spectra. Thus, taking the following boundary value problems

$$-y'' + q(x)y = s^2\alpha^2 y; \quad (1')$$

$$y'(0) - hy(0); \quad y'(a) + Hy(a) = 0 \quad (2')$$

and

$$-y'' + q(x)y = s^2\alpha^2 y, \quad (1'')$$

$$y'(0) - h_1y(0) = 0; \quad y'(a) + H(y(a) = 0 \quad (2'')$$

where $h, h_1$ and $H$ are real numbers such that $h \neq h_1$.

Denote by $\varphi(x,s)$ and $Z(x,s)$ the solutions of equation (1') with the initial conditions $\varphi(0,s) = 1$, $\varphi'(0,s) = h$ and $Z(0,s) = 1$, $Z'(0,s) = h_1$ respectively.

Then

$$M(s) = \frac{Z'(a,s) + HZ(a,s)}{\varphi'(a,s) + Hz(a,s)},$$

which is called Weyl’s function of the problems (1') – (2') and (1'') – (2'').

Since

$$\varphi(x,s) = \frac{1}{2is} [W(s)F(x,-s) - W(-s)F(x,s)]$$

and

$$Z(x,s) = \frac{1}{2is} [W_1(s)F(x,s) - W_1(-s)F(x,s)],$$

where

$$W_1(s) = F'(0,s) - h_1F(0,s),$$

$$M(s) = \frac{S_1(s)F'(a,s) - F'(a,-s) + H[S_1(s)F(a,s) - F(a,-s)]}{F'(a,-s) - S(s)F'(a,s) + H[F(a,-s) - S(s)F(a,s)],}$$

and

$$M(s) = \frac{S_1(s)F'(a,s) - F'(a,-s) + H[S_1(s)F(a,s) - F(a,-s)]}{F'(a,-s) - S(s)F'(a,s) + H[F(a,-s) - S(s)F(a,s)],}$$
where
\[ S_1(s) = \frac{W_1(-s)}{W_1(s)}. \]

The function \( M(s) \) is meromorphic such that its poles and zeros coincide with the eigenvalues of the problems \((1') - (2')\) and \((1'') - (2'')\) respectively. Since the functions \( S(s), F(a,s) \) and \( F'(a,s) \) are defined, thus \( M(s) \) is uniquely defined by this way.

We set up the function
\[ \sigma(s) = \lim_{\zeta \to 0} \frac{1}{a} \int_0^s \text{Im} M(s + i\zeta) ds. \]

Hence the function \( q \) is uniquely defined by two spectra on \([0,a]\) from the work of Gelfand-Levitan-Gasymov [7, 9]. Finally, we conclude that equation (1) can be reconstructed on the interval \([0, \infty)\) and this completes the proof of the theorem. 

References


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