A COUNTEREXAMPLE TO THE GENERALIZED BLOCH PRINCIPLE

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Abstract. If \( f \) is a meromorphic function in the plane such that \( f \) and a derivative \( f^{(k)} \) share three distinct values \( a_1, a_2, a_3 \in \mathbb{C} \), then \( f \equiv f^{(k)} \). If the proposition holds for \( k = 1 \) and each \( f \) in family \( F \) of meromorphic functions in a domain \( G \subset \mathbb{C} \), then \( F \) is normal in \( G \). The authors show that the same conclusion is false for \( k \geq 2 \).

According to Bloch’s principle a condition which reduces a meromorphic function in the plane to a constant makes a family of meromorphic functions in a domain \( G \subset \mathbb{C} \) normal. Although the principle is false in general (compare [4]), a rigorous version has been formulated (compare [9]). Indeed starting with Montel (compare [5][p. 54, p. 74]) several authors (compare [1], [6], [8]) have successfully proved normality criteria starting from Picard type theorems, i.e. from conditions for meromorphic functions in the plane which are only satisfied by constants or special classes of functions. In the last case normality sometimes couldn’t be proved for the family \( F \) itself, but for an auxiliary family constructed by the elements of \( F \) (compare [6], Theorem 4.2).

In [7] one of the authors studied families \( F \) of meromorphic functions in a domain \( G \), for which each \( f \in F \) shares three values \( a_1, a_2, a_3 \in \mathbb{C} \) with its derivative. Mues and Steinmetz ([3]) had proved

**Theorem 1.** Let \( f \) be a nonconstant meromorphic function in the plane and \( a_1, a_2, a_3 \) distinct complex numbers. If \( f \) and \( f' \) share \( a_1, a_2, a_3 \), then \( f \equiv f' \).

The analogous result for normal families is given by (compare [7]):

**Theorem 2.** Let \( F \) be a family of meromorphic functions in a domain \( G \subset \mathbb{C} \) and \( a_1, a_2, a_3 \) distinct complex numbers. If \( f \) and \( f' \) share \( a_1, a_2, a_3 \) for each \( f \in F \), then \( F \) is normal in \( G \).

The authors expanded Theorem 1 to the \( k \)th derivative (compare [2]):

**Theorem 3.** Let \( f \) be a nonconstant meromorphic function in the plane, \( a_1, a_2, a_3 \) distinct complex numbers and \( k \geq 2 \). If \( f \) and \( f^{(k)} \) share \( a_1, a_2, a_3 \) then \( f \equiv f^{(k)} \).

The question arises, whether Theorem 2 holds for \( f^{(k)} \) instead of \( f' \) (\( k \geq 2 \)). In the following we consider \( a_1, a_2, a_3 \in \mathbb{C} \), \( k \geq 2 \) and \( \alpha_k = e^{\frac{2\pi i}{k}} \). If \( \mathbb{D} \) is the unit disk, define \( \wp_k : \mathbb{D} \to \mathbb{C} \), \( \wp_k(z) = e^z - e^{\alpha_k z} \), and \( f_n : \mathbb{D} \to \mathbb{C} \), \( f_n(z) = n \cdot \wp_k(z) \). Then \( f_n \equiv f_n^{(k)} \), which implies that \( f_n \) and \( f_n^{(k)} \) share \( a_1, a_2, a_3 \). Since \( \wp_k(z) = 0 \) iff \( z = 0 \), we get \( f_n(z) \to \infty \) for \( z \neq 0 \). Then \( f_n(0) = 0 \) implies that the sequence \( \{f_n\}_{n \in \mathbb{N}} \) is not normal at \( 0 \).

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Let $F$ be a family of meromorphic functions in a domain $G$, such that $f$ and $f^{(k)}$ share $a_1, a_2, a_3 \in \mathbb{C}$ for each $f \in F$. The above counterexample induces the following question:

Is the family $\{ f - f^{(k)} \mid f \in F \}$ normal in $G$ or does the alternative hold that either $\{ f - f^{(k)} \mid f \in F \}$ or the family $F$ itself is normal in $G$?

We are able to construct a sequence of analytic functions such that both families are not normal. The function $f_n(z) = n \cdot \varphi_k(z)$ has exactly one zero in $\mathbb{D}$ and $f_n(z) \to \infty$ on $|z| = r > 0$. Then Rouché's theorem implies that the equation $f_n(z) = a_j (j = 1, 2, 3)$ has exactly one solution $z_j^{(n)}$ in $\mathbb{D}$, if $n$ is sufficiently large, and $z_j^{(n)} \to 0$. Define

$$h_n(z) = \prod_{j=1}^{3} (z - z_j^{(n)})^{k+1} \quad \text{and} \quad g_n(z) = f_n(z) + \sqrt{n} \cdot h_n(z).$$

Rouché's theorem again yields that $g_n(z) = a_j$ has the only solution $z_j^{(n)}$ and the same holds for $g_n^{(k)}$. Therefore $g_n(z)$ and $g_n^{(k)}(z)$ share $a_1, a_2, a_3$, but $\{g_n\}$ is not normal at 0, because $g_n(z) \to \infty$ for $z \neq 0$ and $g_n(z_j^{(n)}) = a_j (z_j^{(n)} \to 0)$. The difference

$$g_n(z) - g_n^{(k)}(z) = \sqrt{n}(h_n(z) - h_n^{(k)}(z))$$

tends to infinity for $|z| = r > 0$, since

$$h_n(z) - h_n^{(k)}(z) \Rightarrow z^{3k+3} - (3k + 3) \cdots (2k + 4)z^{2k+3},$$

which is different from 0 for $z \neq 0$ and $|z| < 1$. Then $g_n(z_j^{(n)}) - g_n^{(k)}(z_j^{(n)}) = 0$ implies that $\{g_n(z) - g_n^{(k)}\}$ is not normal at 0.

References

