NON ABSOLUTE INTEGRATION USING VITALI COVERS

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Abstract. In this note, we develop nonabsolute integration theory using Vitali covers and Riemann sums in which the partitioning property may not be satisfied.

In [2], we have considered absolute integration using Vitali covers, in this note, we shall consider the nonabsolute case. In the Henstock theory of integration, we use Riemann sums to define an integral. An immediate question is to prove the existence of partitions of a given interval. An advantage of using Vitali covers is that even when the partitioning property is absent, we still can define an integral. The \( B \)-variational integral defined by Thomson [6] and the proximal integral defined by Sarkhel [4] are integrals of this sort. The \( B \)-variational integral is a generalization of the Henstock integral and the AP integral. In this note, we shall prove a convergence theorem for the \( B \)-variational integral, which generalizes the controlled convergence theorems for the Henstock integral and the AP integral. Also, we shall give examples in which the theory is still valid without having to prove the partitioning property.

1. Nonabsolute Integrals

The terminology used in this paper follows mainly Thomson’s papers [6].

Let \( R \) be the set of real numbers and \( I \) be the set of all intervals in \( R \). An element \((I, x) \in I \times R \) is called an interval-point pair. The point \( x \) is called the associated point of the interval \( I \). Let \( \beta \) be a collection of interval-point pairs. Then \( \beta \) is said to be a Vitali cover of a set \( E \subset R \) if for each \( \varepsilon > 0 \) and any \( x \) in \( E \) there is an interval-point pair \((I, x) \in \beta \) such that \( x \in I \) and the length \(|I|\) of \( I \) is less than \( \varepsilon \). Let \( B \) be a collection of Vitali covers of \([a, b] \). Then \( B \) is filtering if \( \beta_1, \beta_2 \in B \) then there is \( \beta_3 \in B \) with \( \beta_3 \subset \beta_1 \cap \beta_2 \). Let \( D \subset \beta, \beta \in B \). For brevity, we write \( D = \{(I, x)\} \) where \((I, x)\) denotes a typical interval-point pair in \( D \). Then \( D \) is said to be a partial \( \beta \)-partition of \([a, b] \) if \( \{I; (I, x) \in D\} \) is a collection of nonoverlapping subintervals of \([a, b] \). A partial \( \beta \)-partition \( D = \{(I, x)\} \) of \([a, b] \) is a \( \beta \)-partition of \([a, b] \) if \( \cup \{I; (I, x) \in D\} = [a, b] \).

Throughout this paper, a collection of Vitali covers of \([a, b] \) is always denoted by \( B \). We always assume that \( B \) is filtering. \( (D) \sum \) denotes the sum over \( D = \{(I, x)\} \) and \(|I|\) denotes the length of an interval \( I \).

Let \( \beta \in B \) and \( E \subset [a, b] \). We write

\[ \beta[E] = \{(I, x) \in \beta; x \in E\}. \]

A collection \( B \) is said to have local character if for each \( x \in [a, b] \), a \( \beta_x \in B \) is given then there is a \( \beta \in B \) for which \( \beta \subset \bigcup_{x \in [a, b]} \beta_x[\{x\}] \) i.e., for which \( \beta[\{x\}] \subset \beta_x \) for all \( x \in [a, b] \).

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An interval function in $[a,b]$ is a mapping from $I$ and having real values. An interval function in $[a,b]$ is additive if $F(I \cup J) = F(I) + F(J)$ for any pair of nonoverlapping subintervals $I$ and $J$ of $[a,b]$ for which $I \cup J$ is an interval. A function $h : I \times [a,b] \to R$ is called an interval-point function. We consider point functions $f : [a,b] \to R$ and interval functions $F : I \to R$ as special cases of interval-point functions by agreeing that $f(I, x) = f(x)$ and $F(I, x) = F(I)$.

Let $h : I \times [a,b] \to R$ be an interval-point function and $\beta \in B$. We write

$$V(h, \beta) = \sup \left\{ (D) \sum |h(I, x)| \mid D = \{(I, x)\} \text{ a partial } \beta \text{-partition of } [a,b] \right\},$$

and refer to $V(h, \beta)$ as the variation of $h$ over $\beta$. The variation of $h$ over $B$ is defined as

$$V(h, B) = \inf \{V(h, \beta) \mid \beta \in B\}.$$

Let $D_1 = \{(I, x)\}, D_2 = \{(J, y)\}$ be two partial $\beta$-partitions. Then $D_2$ is said to be finer than $D_1$, denoted by $D_2 < D_1$, if for each $J$ in $D_2$, there exists $I$ in $D_1$ such that $J \subseteq I$.

An additive interval function $F$ is said to be $AC_B^*(X)$, where $X \subseteq [a,b]$, if for every $\varepsilon > 0$, there exists $\beta \in B$ and $\eta > 0$ such that for any two partial $\beta$-partitions $D_1, D_2$ with associated points in $X$ and $D_2 \leq D_1$ satisfying $(D_1 \setminus D_2) \sum |I| < \eta$, we have

$$\left| (D_1 \setminus D_2) \sum F(I) \right| < \varepsilon,$$

where $(D_1 \setminus D_2) \sum$ denotes $(D_1) \sum - (D_2) \sum$. Here $D_2$ may be void.

In the above definition, if we only consider one partial $\beta$-partition $D_1$, then $F$ is said to be $AC_B^*(X)$.

An additive interval function $F$ is said to be $ACG_B^*$ if $[a,b] = \bigcup_{i=1}^{\infty} X_i$, so that $F$ is $AC_B^*(X_i)$ for each $i$. Also, $ACG_B^*$ can be similarly defined.

A collection $B$ of Vitali covers of $[a,b]$ is said to be complete if for any $ACG_B^*$ additive interval function $F$ in $[a,b]$ with $V(F,B) = 0$, we have $F \equiv 0$.

Let $\beta \in B$ and $\delta(x) > 0$ on $[a,b]$. Then $\beta$ is said to be $\delta$-fine if $x \in I \subseteq (x - \delta(x), x + \delta(x))$ whenever $(I, x) \in \beta$. A collection $B$ is said to have the $\delta$-fine property if for every $\delta(x) > 0$ on $[a,b]$, there exists $\beta \in B$, which is $\delta$-fine. If $B$ has the $\delta$-fine property and is filtering, then for every $\beta \in B$ and for every $\delta(x) > 0$ on $[a,b]$, there exists $\beta_1 \in B$ such that $\beta_1 \subseteq \beta$ and $\beta_1$ is $\delta$-fine. Furthermore if $G$ is open, then there exists $\beta \in B$ such that $I \subseteq G$ whenever $x \in G$ and $(I, x) \in \beta$.

In this note, we always assume that $B$ is filtering and complete. Furthermore, we also assume that $B$ has the $\delta$-fine property.

**Definition 1 [6, p. 381]**. A measurable function $f$ defined on $[a,b]$ is said to be $B$-integrable if there exists an additive interval function $F$ in $[a,b]$ such that

$$V(F - h; B) = 0,$$

where $h(I, x) = f(x)|I|$, i.e., for every $\varepsilon > 0$, there exists $\beta \in B$ such that

$$(D) \sum |F(I) - f(x)|I| < \varepsilon,$$
whenever $D = \{(I,x)\}$ is a partial $\beta$-partition of $[a,b]$.

Denote $F(u,v) = \int_u^v f$. Where $F$ is called the primitive of $f$.

We remark that the above definition given in [6, p. 381] is more general. In [6], we do not assume that $f$ is measurable and $\mathcal{B}$ is a collection of Vitali covers. However, with these imposed conditions, we can prove a better convergence theorem, and most of known integrals satisfy these conditions. Three examples will be given in the last section.

**Theorem 1.** If $f$ is $\mathcal{B}$-integrable on $[a,b]$ with primitive $F$, then $F$ is $ACG_B^{**}$.

**Proof.** If $f$ is $\mathcal{B}$-integrable on $[a,b]$, then, by definition, $f$ is measurable on $[a,b]$. Let $X = \{x; |f(x)| \leq N\}$. Let $f_X(x) = f(x)$, when $x \in X$ and 0 otherwise. Then $f_X$ is measurable and bounded. Hence $f_X$ is McShane integrable on $[a,b]$, see [1, p. 108]. In other words, for every $\varepsilon > 0$, there exists $\delta(x) > 0$ such that for every $\delta$-fine McShane partition $D = \{(I,x)\}$, we have

$$(D) \sum |F_X(I) - f_X(x)|I| < \varepsilon,$$

where $F_X$ is the primitive of $f_X$. Recall that a partition $D = \{(I,x)\}$ is said to be $\delta$-fine McShane partition if $[a,b]$ is the union of intervals $I, I \subset (x - \delta(x), x + \delta(x))$ and $x$ may not belong to $I$.

On the other hand, there exists $\beta \in \mathcal{B}$ such that

$$(D) \sum |F(I) - f(x)|I| < \varepsilon,$$

whenever $D = \{(I,x)\}$ is a partial $\beta$-partition of $[a,b]$. Let $\beta_1 \in \mathcal{B}$ such that $\beta_1 \subset \beta$ and $\beta_1$ is $\delta$-fine. It is possible, since $\mathcal{B}$ is filtering and has the $\delta$-fine property. Now take any two partial $\beta_1$-partitions $D_1 = \{(I,x)\}, D_2 = \{(J,y)\}$ with associated points in $X$ and $D_2 \leq D_1$ such that $(D_1 \backslash D_2) \sum |I'| \leq \eta$. When $I'$ is taken from $D_1 \backslash D_2$, we use the associated point in $D_1$. Then we have

$$\left| (D_1 \backslash D_2) \sum F(I') \right| \leq (D_1) \sum |F(I) - f(x)|I|$$

$$+ (D_1) \sum |F_X(I) - f_X(x)|I|$$

$$+ (D_2) \sum |F(J) - f(y)|J|$$

$$+ (D_2) \sum |F_X(J) - f_X(y)|J|$$

$$+ (D_1 \backslash D_2) \sum |F_X(I') - f_X(x)||I'|$$

$$+ (D_1 \backslash D_2) \sum |f_X(x)||I'|$$

$$< \varepsilon + \varepsilon + \varepsilon + \varepsilon + \varepsilon + N\eta.$$

Thus $F$ is $ACG_B^{**}(X)$. Hence $F$ is $ACG_B^{**}$.

**Theorem 2.** The primitive $F$ of a $\mathcal{B}$-integrable function is uniquely determined.

**Proof.** It follows from Theorem 1 and the fact that $\mathcal{B}$ is complete.
2. Convergence Theorem

In this section, we shall prove the following controlled convergence theorem.

**Theorem 3.** In addition to \( \mathcal{B} \) being filtering, complete, and having the \( \delta \)-fine property, let \( \mathcal{B} \) have the local character. If \( \{f_n\} \) is a sequence of \( \mathcal{B} \)-integrable functions on \([a, b]\) satisfying the following properties:

(i) \( f_n \) converges to \( f \) almost everywhere on \([a, b]\) as \( n \to \infty \);

(ii) the primitive \( F_n \) of \( f_n \) are uniformly \( ACG^{**} \) on \([a, b]\);

and

(iii) the primitive \( F_n \) converges to \( F \) pointwise on \([a, b]\) as \( n \to \infty \),

then \( f \) is \( \mathcal{B} \)-integrable on \([a, b]\) and

\[
\int_a^b f_n \longrightarrow \int_a^b f.
\]

We shall use the following Lemmas.

**Lemma 4.** Let \( f \) be \( \mathcal{B} \)-integrable on \([a, b]\) with primitive \( F \). If \( F \) is \( AC^{**}_B(X) \), where \( X \) is a closed subset of \([a, b]\), then \( f_X \) is Lebesgue integrable on \([a, b]\).

**Proof.** Let \( \beta \in \mathcal{B} \). Since \( F \) is \( AC^{**}_B(X) \), we can define

\[
H_\beta = \inf_{\eta_0 > 0} \sup_{(D) \in \mathcal{B}} (D) \sum |I| > |X| - \eta_0,
\]

where the supremum is over all partial \( \beta \)-partitions \( D = \{(I, x)\} \) with \( x \in X \) and \((D) \sum |I| > |X| - \eta_0 \). Note that \( \beta \) is a Vitali cover of \([a, b]\). Let \( \varepsilon > 0 \). Choose \( \beta \in \mathcal{B} \) such that \( 0 \leq H_\beta - H < \varepsilon \). Next, choose an open set \( G \supset X \) with \( |G \setminus X| < \eta/5 \), where \( \eta \) comes from the definition of \( AC^{**}_B(X) \) with given \( \varepsilon > 0 \). We may choose \( \beta \) such that \( \beta \subset \beta_0 \) where \( \beta_0 \) is given in the definition of \( AC^{**}_B(X) \) with given \( \varepsilon > 0 \). Furthermore, since \( \mathcal{B} \) has the \( \delta \)-fine property, we may assume that \( I \subset G \) when \((I, x) \in \beta \) and \( x \in X \). Now choose \( \eta_1 < \eta/5 \) and a fixed partial \( \beta \)-partitions \( D_0 = \{(I, x)\} \) with \( x \in X \) and \((D_0) \sum |I| > |X| - \eta_1 \) such that

\[
H_\beta - \varepsilon \leq (D_0) \sum |I| \leq H_\beta + \varepsilon.
\]

Since \( f \) is \( \mathcal{B} \)-integrable on \([a, b]\), there is \( \beta_1 \in \mathcal{B} \) such that for any partial \( \beta_1 \)-partition \( D = \{(I, x)\} \) of \([a, b]\), we have

\[
(D) \sum |f(x)| |I| - F(I)| < \varepsilon.
\]

We may choose \( \beta_1 \subset \beta_0 \). Furthermore since \( \mathcal{B} \) has the \( \delta \)-fine property, we may assume that if \((I, x) \in \beta_1 \) and \( x \in J^0 \) the interior of \( J \) for some \((J, y) \in D_0 \), then \( I \subset J \); if \((I, x) \in \beta_1 \), \( x \not\in J \) for all \((J, y) \in D_0 \), then \( I \cap J = \emptyset \) for all \((J, y) \in D_0 \).
Note that we may assume that the total length of intervals $I$ of any partial $\beta_1$-partition with $(I, x) \in \beta_1$, $x \in J \setminus J^0$ for some $(J, y) \in D_0$ is less than $\eta/5$. We also assume that $X \cap I = \emptyset$ when its associated point $x \notin X$. Let $D = \{(I, x)\}$ be any partial $\beta_1$-partition of $[a, b]$ with

$$(D) \sum |I| > b - a - \eta/5. \quad (*)$$

Let

$$
D_1 = \{(I, x) \in D; x \in X\},
D_2 = \{(I, x) \in D_1; x \in J \text{ for some } (J, y) \in D_0\}
$$

and

$$D_3 = D_1 \setminus D_2.$$

Note that $(D_1) \sum |I| > |X| - \eta/5$ since $X \cap I = \emptyset$ when its associated point $x \notin X$ and condition $(*)$ holds. Furthermore

$$(D_3) \sum |I| \leq |G \setminus X| + |X \setminus D_0|$$

$$< \eta/5 + \eta_1$$

$$< \frac{2}{5} \eta,$$

where $|X \setminus D_0|$ means $|X \cup \{(I, x) \in D_0\}|$, and

$$(D_0) \sum |I| - (D_2) \sum |I| \leq |G \setminus X| + |X \setminus D_2| + \sum_{x \in J \setminus J^0, (J, y) \in D_0} |I|$$

$$\leq |G \setminus X| + |X \setminus D_1| + (D_3) \sum |I| + \eta/5$$

$$< \eta/5 + \eta/5 + 2\eta/5 + \eta/5.$$

Then, in view of $(D) \sum f_X(x)|I| = (D_1) \sum f(x)|I|$, we have

$$\left| (D) \sum f_X(x)|I| - H \right| \leq \left| (D_1) \sum \{f(x)|I| - F(I)\} \right|$$

$$+ \left| (D_0) \sum F(I) - H \beta \right|$$

$$+ \left| (D_0) \sum F(I) - (D_2) \sum F(I) \right|$$

$$+ \left| H \beta - H \right|$$

$$+ \left| (D_1 \setminus D_2) \sum F(I) \right|$$

$$< 5\varepsilon.$$

Hence $f_X$ is Lebesgue integrable on $[a, b]$, by Theorem 9 in [2].
Lemma 5. Let \( f_n \) be a sequence of \( B \)-integrable functions on \([a, b]\) with primitives \( F_n \). If \( F_n \) are uniformly \( AC^{**}(X) \), where \( X \) is closed, then \( f_{n,x} \) is a sequence of \( B \)-integrable functions on \([a, b]\) with primitives \( F_{n,x} \) where \( f_{n,x}(x) = f_n(x) \) when \( x \in X \) and zero otherwise and the following condition is satisfied: for every \( \varepsilon > 0 \), there exists \( \beta \in B \), independent of \( n \), such that for any partial \( \beta \)-partition \( D = \{(I, x)\} \) of \([a, b]\) with \( x \in X \), we have

\[
\left| (D) \sum (F_{n,x}(I) - F_n(I)) \right| < \varepsilon,
\]

for all \( n \).

Proof. The first part follows from Lemma 4. Now we prove the second part. Since \( F_n \) are \( AC^{**}(X) \) uniformly, for every \( \varepsilon > 0 \) there exist \( \beta \in B \) and \( \eta > 0 \), both independent of \( n \), such that the rest of the condition for \( AC^{**}(X) \) holds. For each \( n \), there exists \( \beta_n \in B \) with \( \beta_n \subset \beta \) such that for any partial \( \beta_n \)-partition \( D = \{(I, x)\} \) of \([a, b]\), we have

\[
(D) \sum |F_n(I) - f_n(x)|I| < \varepsilon,
\]

and

\[
(D) \sum |F_{n,x}(I) - f_{n,x}(x)|I| < \varepsilon.
\]

We may assume that \( I \cap X = \emptyset \) when its associated point \( x \notin X \) and \( I \subset G \) if \( x \in X \), where \( G \supset X, |G \setminus X| < \eta_1/2 \) and \( G \) is open, where \( \eta_1 < \eta \) and \( \eta_1 \) comes from the definition of uniform absolute continuity of \( F_{n,x} \) on \([a, b]\) with given \( \varepsilon > 0 \). Let \( D = \{(J, y)\} \) be any partial \( \beta \)-partition of \([a, b]\) with \( y \in X \). Let \( D_1 = \{(I, x)\} \) be a partial \( \beta_n \)-partition of \([a, b]\) such that each \( I \) is in \( J \) for some \((J, y) \in D \) and \(
|D| < D_1 |J| - D_1 |J| < \eta_1/2\). Split \( D_1 \) into \( D_2 \) and \( D_3 \) so that \( D_2 \) contains the intervals with associated points in \( X \) and \( D_3 \) otherwise. Note that

\[
(D \setminus D_1) \sum |I| < \eta_1/2,
\]

\[
(D \setminus D_2) \sum |I| \leq (D \setminus D_1) \sum |I| + (D_1 \setminus D_2) \sum |I|
\]

\[
< \eta_1/2 + |G \setminus X|
\]

\[
< \eta_1/2 + \eta_1/2
\]

\[
< \eta,
\]

and

\[
(D_1) \sum f_{n,x}(x)|I| = (D_2) \sum f_{n,x}(x)|I|.
\]

Then

\[
\left| (D) \sum (F_{n,x}(I) - F_n(I)) \right| \leq \left| (D_1) \sum (F_{n,x}(I) - f_{n,x}(x)|I|) \right|
\]

\[
+ \left| (D_2) \sum f_{n,x}(x)|I| - F_n(I) \right|
\]

\[
+ \left| (D \setminus D_1) \sum F_{n,x}(I) \right| + \left| (D \setminus D_2) \sum F_n(I) \right|
\]

\[
< \varepsilon + \varepsilon + \varepsilon + \varepsilon.
\]
Lemma 6. If the conditions in Lemma 5 are satisfied, then \( F_{n,x} \) are uniformly absolutely continuous on \([a, b]\).

**Proof.** By Lemma 5, the sequence \( F_{n,x} \) are uniformly \( AC^*_{B}(X) \). Note that we may choose \( \beta \in \mathcal{B} \) such that \( I \cap X = \emptyset \) when \((I, x) \in \beta \) with its associated point \( x \notin X \), since \( \mathcal{B} \) has the \( \delta \)-fine property. Thus \( F_{n,x}(I) = 0 \), when \((I, x) \in \beta \), \( x \notin X \). Consequently, the sequence \( F_{n,x} \) are uniformly \( AC^*_{B}[a, b] \). By Lemma 4, each \( f_{n,x} \) is Lebesgue integrable on \([a, b]\). Let \( \varepsilon > 0 \). Let \( \eta > 0 \) and \( \beta \in \mathcal{B} \) (both independent of \( n \)) be given as in the definition of uniformly \( AC^*_{B}[a, b] \) with the given \( \varepsilon > 0 \). Let \( \{[u_i, v_i]\}_{i=1}^{N} \) be a finite sequence of nonoverlapping subintervals of \([a, b]\) with \( \sum_{i=1}^{N} |u_i - v_i| < \eta \). For each \( n \), the function \( F_{n,x} \) is absolutely continuous on \([a, b]\) and \( \beta \) is a Vitali cover of \([a, b]\); thus there exists a finite sequence of nonoverlapping intervals \( \{[x_{n,j}, y_{n,j}]\}_{j=1}^{p(n)} \) from \( \beta \) with \( \bigcup_{j=1}^{p(n)} [x_{n,j}, y_{n,j}] \subset \bigcup_{i=1}^{N} [u_i, v_i] \) such that

\[
\left| \sum_{i=1}^{N} F_{n,x}(u_i, v_i) - \sum_{j=1}^{p(n)} F_{n,x}(x_{n,j}, y_{n,j}) \right| < \varepsilon.
\]

Hence

\[
\left| \sum_{i=1}^{N} F_{n,x}(u_i, v_i) \right| \leq \left| \sum_{i=1}^{N} F_{n,x}(u_i, v_i) - \sum_{j=1}^{p(n)} F_{n,x}(x_{n,j}, y_{n,j}) \right| + \left| \sum_{j=1}^{p(n)} F_{n,x}(x_{n,j}, y_{n,j}) \right| < \varepsilon + \varepsilon, \quad \text{for all } n.
\]

Lemma 7. If the conditions in Lemma 5 are satisfied, and, in addition, \( f_n \) converges to \( f \) almost everywhere on \([a, b]\), then \( f_x \) is Lebesgue integrable on \([a, b]\) and for every \( \varepsilon > 0 \), there is a positive integer \( N \) such that for any partial partition \( D = \{I\} \) of \([a, b]\), we have

\[
\left| (D) \sum \{F_{n,x}(I) - F_X(I)\} \right| < \varepsilon.
\]

whenever \( n \geq N \).

**Proof.** It follows from Lemma 6 and Vitali’s convergence theorem (see [1]) for Lebesgue integrals.

Lemma 8. If the conditions in Lemma 7 are satisfied, and, in addition, \( F_n \) converges to \( F \) pointwise on \([a, b]\), then for every \( \varepsilon > 0 \), there exists \( \beta \in \mathcal{B} \) such that for any partial \( \beta \)-partition \( D = \{(I, x)\} \) of \([a, b]\) with \( x \in X \), we have

\[
\left| (D) \sum \{F_X(I) - F(I)\} \right| < \varepsilon.
\]

**Proof.** It follows from Lemmas 5 and 7.
Lemma 9. In addition to $B$ being filtering, complete and have the $\delta$-fine property, let $B$ have the local character. If $\{f_n\}$ is a sequence of $B$-integrable functions on $[a,b]$ satisfying the condition (ii) of Theorem 3, then the following strong Lusin condition holds: for every $Z \subset [a,b]$ of measure zero and for every $\varepsilon > 0$, there exists $\beta \in B$ such that for any partial $\beta$-partition $D = \{(I,x)\}$ of $[a,b]$ with $x \in Z$ we have

$$\left| \left( D \right) \sum F_n(I) \right| < \varepsilon$$

for all $n$.

Proof. Let $F_n$ be uniformly $AC^*_B(X_i)$, where $[a,b] = \bigcup_i X_i$. Let $Z \subset [a,b]$ with $|Z| = 0$. Let $S_i = Z \cap X_i$. Then $|S_i| = 0$. Let $\varepsilon > 0$ and $G_i$ an open set such that $S_i \subset G_i$ and $|G_i \setminus S_i| < \eta_i$ where $\eta_i$ comes from the definition of $AC^*_B(X_i)$ for the given $\varepsilon 2^{-i}$. Let $\beta_i \in B$ be the corresponding Vitali cover in the definition of $AC^*_B(X_i)$. Since $B$ has the $\delta$-fine property, we may assume that $I \subset G_i$ when $(I,x) \in \beta_i$ with $x \in S_i$. Since $B$ has local character, there exists $\beta \in B$ such that $\beta[\{x\}] \subset \beta_i$ if $x \in X_i \setminus X_{i-1}$, $i = 1,2,\ldots$. Where $X_0 = \emptyset$. Then for any partial $\beta$-partition $D = \{(I,x)\}$ of $[a,b]$ with $x \in Z$, we have

$$\left| \left( D \right) \sum F_n(I) \right| \leq \sum_i \varepsilon 2^{-i} = \varepsilon.$$

Proof of Theorem 3. Let $F_n$ be uniformly $AC^*_B(X_i)$, where $[a,b] = \bigcup_i X_i$. We may assume that $X_i \subset X_{i+1}$ for all $i$. Let $Y_i$ be a closed subset of $X_i$ such that $|Z| = 0$ where $Z = [a,b] \setminus \bigcup_i Y_i$. Let $\varepsilon > 0$. First, by Lemma 7, there exists a positive integer $n = n(i,j)$ such that for any partial partition $D = \{I\}$ of $[a,b]$, we have

$$\left| \left( D \right) \sum \{F_n, Y_i \} - F_{Y_i}(I) \right| < \varepsilon 2^{-i-j}.$$

We may assume that for each $i$, $\{n(i,j)\}$ is strictly increasing and $\{n(i,j)\}$ is a subsequence of $\{n(i-1,j)\}$. For each $x \in Y_i$, there exists $m(x) = n(j,j)$, for some $j > i$ such that

$$|f_m(x) - f(x)| < \varepsilon.$$

We may assume that $f_n(x)$ converges to $f(x)$ pointwise, for each $x \in [a,b]$ and $f_n(x) = f(x) = 0$ for all $n$ and all $x \in Z$.

There exists $\beta_n \in B$ such that for any partial $\beta_n$-partition $D = \{(I,x)\}$ of $[a,b]$, we have

$$\left| \left( D \right) \sum |F_n(I) - f_n(x)|I| \right| < \varepsilon 2^{-n}.$$

By Lemmas 5, 8, 9, and the fact that $B$ has local character, there exists $\beta \in B$, independent of $n$, such that for any partial $\beta$-partition $D = \{(I,x)\}$ of $[a,b]$ with $x \in W_i$ where $W_i = Y_i \setminus Y_{i-1}$, $Y_0 = \emptyset$, we have

$$\left| \left( D \right) \sum \{F_n, Y_i \} - F_n(I) \right| < \varepsilon 2^{-i}.$$

for all $n$ and

$$\left| \left( D \right) \sum \{F, Y_i \} - F(I) \right| < \varepsilon 2^{-i}.$$
Furthermore, for any partial $\beta$-partition $D = \{(I, x)\}$ of $[a, b]$ with $x \in Z$, we have

$$\left| (D) \sum F_n(I) \right| < \varepsilon$$

for all and

$$\left| (D) \sum F(I) \right| < \varepsilon.$$

Since $B$ has local character and filtering, we may further assume that $\beta([x]) \subset \beta_m(x)$. Hence if $(I, x) \in \beta$ then $(I, x) \in \beta_m(x)$.

Let $D = \{(I, x)\}$ be any partial $\beta$-partition of $[a, b]$. Let $D_2$ be the subset of $D$ such that $x \in Z$ and $D_1 = D \setminus D_2$. Then

$$\left| (D) \sum \{f(x)|I| - F(I)\} \right| \leq \left| (D_2) \sum F(I) \right| + \left| (D_2) \sum f(x)|I| \right|$$

$$+ \left| (D_1) \sum \{f(x)|I| - f_m(x)(x)|I|\} \right|$$

$$+ \left| (D_1) \sum \{F_m(x)(I) - f_m(x)(x)|I|\} \right|$$

$$+ \left| (D_1) \sum \sum \{F_m(x)(I) - F_m(x), y_i(I)\} \right|$$

$$+ \left| (D_1) \sum \sum \{F_y(I) - F(I)\} \right|$$

$$< \varepsilon + \varepsilon(b - a) + \sum_{n=1}^{\infty} \varepsilon 2^{-n}$$

$$+ \sum_{i=1}^{\infty} \varepsilon 2^{-i} + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varepsilon 2^{-i-j} + \sum_{i=1}^{\infty} \varepsilon 2^{-i}.$$

Hence $f$ is $B$-integrable on $[a, b]$ and

$$\lim_{n \to \infty} \int_a^b f_n = \int_a^b f.$$

Note that the sixth term on the right-hand side of the above inequality is equal to

$$(D_1) \sum \sum \sum \{F_m(x), y_i(I) - F_y(I)\}.$$

If $m(x) = n(j, j)$ and $x \in W_i$, then $j > i$. Hence $n(j, j) = n(i, k(j))$ for some $k(j)$. Therefore, the above sum is less than $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varepsilon 2^{-i-k(j)}.$
3. Examples

Example (i). Let $\delta(x) > 0$ on $[a, b]$ and

$$\beta_\delta = \{(x, v]; x \in [u, v] \subset (x - \delta(x), x + \delta(x)), x \in [a, b]\}.$$ 

Then $\beta_\delta$ is an open cover and therefore a Vitali cover of $[a, b]$. Let $B_H$ be the collection of all $B_h$. Then $B_H$ is filtering, has the $\delta$-fine property and local character. Furthermore $B_H$ is complete, see [6, Lemma 5.9, p. 112]. Thus $f$ is $B_H$-integrable on $[a, b]$ iff $f$ is Henstock integrable there.

Example (ii). Let $x \in [a, b]$, and $D_x$ be a measurable subset of $[a, b]$ with $x \in D_x$ and of density 1 at $x$. Let $\beta\{D_x\} = \{([u, v], x); x \in [u, v] \subset D_x \text{ and } x \in [a, b]\}$. Then $\beta\{D_x\}$ is a Vitali cover of $[a, b]$. Let $B_{AP}$ be the collection of all $\beta\{D_x\}$. Then $B_{AP}$ is filtering, has the $\delta$-fine property and local character. Furthermore $B_{AP}$ is complete, see [6, Lemma 5.9, p. 112]. Thus $f$ is $B_{AP}$-integrable on $[a, b]$ iff $f$ is $AP$-integrable there. For a definition of the $AP$ integral, see [1; p. 138].

Example (iii). A set $E$ is said to be sparse at $x$ on the right if for every $\epsilon > 0$ there exists $k > 0$ such that every interval $(a, b) \subset (x, x + k)$ with $|a - x| < k|b - x|$, contains at least one point $y$ such that $|E \cap (x, y)| > \epsilon|x - y|$. It is easy to see that a set $E = [x, b]$ with $\delta > 0$ is sparse at $x$. Similarly, we can define a sparse set at $x$ on the left. A set $E$ is said to be sparse at $x$ if $E$ is sparse at $x$ on the right and on the left. A set $E$ is said to be full at $x$ if the complement of $E$ is sparse at $x$. It is obvious that the set $(x - \delta, x + \delta)$ is full at $x$ for any $\delta > 0$. Sarkhel and De have shown that a sparse set at $x$ has lower density zero at $x$ and given an example to show that its upper density at $x$ may be arbitrarily close to 1 [4, p. 30]. Hence a full set at $x$ has an upper density 1 at $x$ and its lower density at $x$ may be arbitrarily close to zero.

Let $D_x$ be a measurable full set with $x \in D_x$ and $\beta\{D_x\} = \{([u, v], x); x \in [u, v] \subset D_x \text{ and } x \in [a, b]\}$. Then $\beta\{D_x\}$ is a Vitali cover of $[a, b]$. In fact let $x \in [a, b]$, we claim that for every $\epsilon > 0$, there exist $u, v \in D_x$ such that $|u - v| < \epsilon$, where $D_x$ is the corresponding full set at $x$ in $\beta\{D_x\}$. If it is not true, then there exists $\epsilon > 0$ and for every $u, v \in D_x$ we have $|u - v| \geq \epsilon$. Hence $D_x$ has density 0 at $x$, and $D_x$ could not be full at $x$. This leads to a contradiction. Let $B_s$ be the collection of all $\beta\{D_x\}$. Note that if $D_x, D_x^1$ are any two measurable full sets at $x$, then $D_x \cap D_x^1$ is a measurable full set at $x$, see [4, Corollary 3.1.1, p. 32]. Hence $B_s$ is filtering. Furthermore, if $\delta(x) > 0$ is given on $[a, b]$, let $\beta_\delta = \{([u, v], x); x \in [u, v] \subset (x - \delta(x), x + \delta(x)), x \in [a, b]\}$. Then $\beta_\delta \in B_s$. Hence $B_s$ has the $\delta$-fine property. Now, if for each $x$, a $\beta_x \in B_s$ is given. Let $D_x$ be the corresponding full set at $x$. Let $\beta \in B_s$ induced by $\{D_x\}$. Then $\beta = \cup\{\beta_x\}; x \in [a, b]\}$. Hence $B_s$ has local character. Next we shall show that $B_s$ is complete by using the following definition and theorem.

Definition 2. A collection $B$ of Vitali covers of $[a, b]$ is said to have the monotonicity property if for any additive interval function $F$ in $[a, b]$ with

$$(D_B F)(x) = \sup_{\beta \in B} \inf_{(I, x) \in \beta} \frac{F(I)}{|I|} \geq 0,$$

we have $F(I) \geq 0$ for any subinterval $I$ of $[a, b]$. 
Theorem 10. Let $\mathcal{B}$ be a collection of Vitali covers of $[a, b]$. If $\mathcal{B}$ has the monotonicity property, then $\mathcal{B}$ is complete.

Proof. Suppose there exists a $ACG^*_{\mathcal{B}}$ additive interval function $F$ in $[a, b]$ with $V(F; [a, b]) = 0$. Then for $\varepsilon > 0$, there exists $\beta \in \mathcal{B}$ such that whenever $D = \{(I, x)\}$ is a partial $\beta$-partition of $[a, b]$, we have

$$\sum |F(I)| < \varepsilon.$$ 

Let $\chi(x) = \sup(D) \sum |F(I)|$ where the supremum is over all partial $\beta$-partitions $D$ of $[a, x]$. Write $\chi(x, y) = \chi(y) - \chi(x)$. Then

$$-\chi(u, v) \leq F(u, v) \leq \chi(u, v),$$

whenever $([u, v], x) \in \beta$. Hence

$$\overline{D}_{\mathcal{B}}(F - \chi)(x) \leq 0 \leq \underline{D}_{\mathcal{B}}(F + \chi)(x),$$

where

$$\underline{D}_{\mathcal{B}}(F + \chi)(x) = \sup_{\beta \in \mathcal{B}} \inf_{([u, v], x) \in \beta} \frac{F(u, v) + \chi(u, v)}{v - u},$$

and

$$\overline{D}_{\mathcal{B}}(F - \chi)(x) = -\underline{D}_{\mathcal{B}}(\chi - F)(x).$$

Note that

$$0 \leq \{F(a, b) + \chi(a, b)\} - \{F(a, b) - \chi(a, b)\} \leq 2\varepsilon.$$

Hence

$$\inf\{H(a, b)\} = \sup\{G(a, b)\},$$

where the infimum is over all additive functions $H$ in $[a, b]$ such that $0 \leq \underline{D}_{\mathcal{B}}H(x)$ for all $x \in [a, b]$ and the supremum is over all additive functions $G$ in $[a, b]$ such that $\overline{D}_{\mathcal{B}}G(x) \leq 0$ for all $x \in [a, b]$. It is so since

$$0 \leq \inf\{H(a, b)\} - \sup\{G(a, b)\} \leq \{F(a, b) + \chi(a, b)\} - \{F(a, b) - \chi(a, b)\} \leq 2\varepsilon.$$

Note that the first inequality in the above holds since $\underline{D}_{\mathcal{B}}(H - G)(x) \geq 0$ for all $x \in [a, b]$ and therefore, by the monotonicity property, $H(I) \geq G(I)$ for any subinterval $I$ of $[a, b]$.

Now we shall prove that $F(a, b) = \inf\{H(a, b)\}$. Indeed,

$$F(a, b) - \chi(a, b) \leq \inf\{H(a, b)\} \leq F(a, b) + \chi(a, b).$$

Thus

$$-\varepsilon \leq -\chi(a, b) \leq \inf\{H(a, b)\} - F(a, b) \leq \chi(a, b) \leq \varepsilon.$$
Therefore
\[ F(a, b) = \inf \{ H(a, b) \}. \]

Similarly we can prove that \( F(a, x) = \inf \{ H(a, x) \} \), for all \( x \in [a, b] \). Note that in view of the monotonicity property again \( \inf \{ H(a, x) \} \leq 0 \leq \sup \{ G(a, x) \} \) for all \( x \in [a, b] \) since \( D_B 0 = 0 \). Thus \( F \equiv 0 \), i.e., \( B \) is complete.

\( B_s \), in Example (iii), has the monotonicity property, see [4, Theorem 4.3 and line -6]. Therefore \( B_s \) is complete.

Given \( \beta \in B_s \), whether there exists a partition \( a = u_0 < u_1 < \cdots < u_n = b \) with \( ([u_{i-1}, u_i], x_i) \in \beta \) and \( x_i \in [u_{i-1}, u_i] \) is an open question, i.e., whether \( B_s \) has the partitioning property is an open question, see [6, p. 135] for the partitioning property involving density conditions. However \( B_s \) is complete, thus we can define the \( B_s \)-integral.

We remark that if \( B \) is complete, then \( B \) has the monotonicity property with some restrictions imposed on the interval function \( F \), see Definition 2 and [6, §6, p. 318].

References


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