WEIGHTS FOR THE SIMPLE Ree GROUPS $^2G_2(q^2)$

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Abstract. This paper is part of a program to study the weight conjecture of J.L. Alperin for finite groups of Lie type. The local structures of radical subgroups of a simple Ree group $^2G_2(q^2)$ are given and the conjecture has been proved for $^2G_2(q^2)$.

Introduction

Let $G$ be a finite group and $r$ a prime. Denote by $O_r(G)$ the largest normal $r$-subgroup of $G$. Let $R$ be an $r$-subgroup of $G$, and let $C$ and $N$ be the centralizer $C_G(R)$ and normalizer $N_G(R)$ of $R$ in $G$, respectively. Following [2], $R$ is a radical $r$-subgroup of $G$ if $R = O_r(N)$. From [2] p. 3, an irreducible character $\varphi$ of $N$ is called a weight character of $G$ if $\varphi$ is trivial on $R$ and in an $r$-block of defect 0 of $N/R$. If such a character $\varphi$ exists, then $R$ is necessarily a radical subgroup of $G$. A pair $(R, \varphi)$ of an $r$-subgroup $R$ and an irreducible character $\varphi$ of $N$ is called a weight if $\varphi$ is a weight character. A weight $(R, \varphi)$ is always identified with its $G$-conjugates. Let $B(\varphi)$ be the $r$-block of $N$ containing $\varphi$. A weight $(R, \varphi)$ is called a $B$-weight for a block $B$ of $G$ if $B = B(\varphi)^G$, that is, $B$ corresponds to $B(\varphi)$ by the Brauer homomorphism. Alperin in [1] conjectured that the number of weights of $G$ should equal the number of irreducible Brauer characters. Moreover, this equality should hold block by block.

The truth of this conjecture has been proved for several classes of groups. In [2–6] Alperin, Fong and the author verified the conjecture for symmetric, general linear, and unitary groups, and for odd-dimensional special orthogonal groups when the characteristic $r$ of the modular representation is odd. Moreover, the numbers of weights for blocks were given for all classical groups except when the defining characteristic of a group is even and $r$ is odd. In [7] and [8] the author verified the conjecture for the Chevalley group $G_2(q)$ and the Steinberg triality group $^3D_4(q)$ for all primes $r$. In [2] Michler and Olsson verified the conjecture for the covering groups of symmetric and alternating groups when $r$ is odd. In addition several equivalent statements were given, say [18]. In this paper we prove the conjecture for the simple Ree groups $^2G_2(q^2)$ for all $r$, where $q^2$ is an odd power of 3. If an $r$-block $B$ of $G$ has a trivial defect group $D$, then both the number of irreducible Brauer characters and the number of $B$-weights are 1 as $N_G(D) = C_G(D) = G$. Moreover, if an $r$-block $B$ of $G$ has a cyclic defect group, then the conjecture for $B$ follows by Theorem 3.8 of [18] and Theorems 8.3 and 9.1 of [11]. The proof in this paper, however, deals with both cyclic and non-cyclic cases. We may always suppose $r$ is different from 3 since the result is known when $r$ is 3 (see [1]).

In Section 1 we describe local structures of radical $r$-subgroups of $^2G_2(q^2)$. In Section 2 we classify the blocks and count numbers of irreducible Brauer characters of blocks. The conjecture is proved in Section 3.

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1. The Radical Subgroups

Let $r$ be a prime different from 3, $q^2 = 3^{2m+1}$ for some $m \geq 0$, and let $G = 2^2G_2(q^2)$. Note that $q$ is not an integer. Thus the Ree group $G$ is simple for $m \geq 1$ and $2^2G_2(3)$ is isomorphic to $PTL_2(8)$. In this section, we shall determine the local structures of radical $r$-subgroups of $G$. For simplicity, we always suppose that $m \geq 1$.

Throughout this paper we shall follow the notation of [16], [17], and [22]. In particular, if $X$ and $Y$ are groups, then $X \rtimes Y$ denotes a semidirect product of $X$ and $Y$ with $X \triangleleft X \times Y$, and $X \circ Y$ or simply $XY$ denotes a central product of $X$ and $Y$. Given a non-negative integer $m$, we denote $D_{2m}$ a dihedral group of order $2m$, and write $\mathbb{Z}_m$ for the cyclic group of order $m$. We also denote by $L_m(q^2)$ the group $PSL_m(q^2)$.

Let $G = 2^2G_2(q^2)$ be the Ree group of type $G_2$. Then the order of $G$ is

$$|G| = q^6(q^2 - 1)(q^2 + \sqrt{3}q + 1)(q^2 - \sqrt{3}q + 1)$$

and $G$ contains 4 conjugacy classes of maximal tori with the representatives $T_1$, $T_2$, $T_3$, and $T_4$. Their structure is

$$T_1 \simeq \mathbb{Z}_{q^2-1} \simeq \langle u \rangle \times \mathbb{Z}_{\frac{1}{2}(q^2-1)},$$
$$T_2 \simeq \langle u \rangle \times \mathbb{Z}_{\frac{1}{2}(q^2+1)} \simeq \langle u \rangle \times \langle u' \rangle \times \mathbb{Z}_{\frac{1}{2}(q^2+1)},$$
$$T_3 \simeq \mathbb{Z}_{q^2+\sqrt{3}q+1},$$
$$T_4 \simeq \mathbb{Z}_{q^2-\sqrt{3}q+1},$$

where $u$ and $u'$ are commuting involutions. Each element $t$ in $\bigcup_{i=1}^4 T_i \setminus \langle u, u' \rangle$ is regular (cf. [16], p. 872), so that $C_G(t) = T_i$ whenever $t \in T_i$. In addition, $G$ has a unique conjugacy class of involutions and a unique class of four-groups, and a Sylow 2-subgroup $S$ of $G$ is elementary abelian of order 8. It follows by Theorem C of [17] that

$$C_G(u) \simeq \mathbb{Z}_2 \times L_2(q^2)$$
$$C_G(\langle u, u' \rangle) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times D_{\frac{1}{2}(q^2+1)},$$

$$C_G(S) = S$$

Moreover, by [22] and Theorem C of [17] (cf. also [16], p. 873)

$$N_G(\langle t_1 \rangle) = T_1 \rtimes \mathbb{Z}_2$$
$$N_G(\langle t_2 \rangle) = T_2 \rtimes \mathbb{Z}_6$$
$$N_G(\langle t_3 \rangle) = T_3 \rtimes \mathbb{Z}_6$$
$$N_G(\langle t_4 \rangle) = T_4 \rtimes \mathbb{Z}_6$$

$$N_G(\langle u \rangle) = C_G(u) = \mathbb{Z}_2 \times L_2(q^2),$$
$$N_G(\langle u, u' \rangle) = (\mathbb{Z}_2 \times \mathbb{Z}_2 \times D_{\frac{1}{2}(q^2+1)}) \rtimes \mathbb{Z}_3,$$
$$N_G(S) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7 \times \mathbb{Z}_3.$$

Suppose that $r$ divides $|G|$. If $r \neq 2$, then a Sylow $r$-subgroup $S$ is a Sylow $r$-subgroup of a maximal torus, so that it is cyclic by (1.2). Thus a radical $r$-subgroup
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$R$ is also cyclic. If $r = 2$, then $S = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, so a radical 2-subgroup $R$ is elementary abelian. In both cases $R \leq C_G(R)$ and $C_G(R) \leq N_G(R)$. Since $O_r(C_G(R))$ is a characteristic subgroup of $C_G(R)$, it follows that $O_r(C_G(R)) \leq O_r(N_G(R))$. But $R$ is radical, so $R = O_r(N_G(R))$, and $O_r(C_G(R)) \leq R$. Similarly, $R \leq O_r(C_G(R))$ as $R \leq C_G(R)$. Thus $R$ is the maximal normal $r$-subgroup of $C_G(R)$.

If $r \neq 2$ and $|R| \neq 1$, then $C_G(R)$ is a maximal torus $T$ of $G$ as a generator of $R$ is regular. Thus $R = O_r(T)$, $R$ is a Sylow $r$-subgroup of $G$, and $N_G(R)$ is given by (1.4). If $r = 2$ and $|R| \neq 1$, then $|R| \in \{2, 4, 8\}$. Since $G$ has a unique class of involutions and a unique class of four-groups, we may suppose that $R \in \{\langle u \rangle, \langle u, u' \rangle, S\}$, where $S$ is a Sylow 2-subgroup of $G$. Conversely, if $R \in \{\langle u \rangle, \langle u, u' \rangle, S\}$, then $R = O_2(C_G(R))$ by (1.3) and $R$ is radical by (1.4). The centralizer and normalizer of $R$ are given by (1.3) and (1.4), respectively. Thus we have proved the following proposition.

Proposition (1A). Let $R$ be a (non-trivial) radical $r$-subgroup of $G = ^2G_2(q^2)$, $N = N_G(R)$, and $C = C_G(R)$. Suppose that $r$ divides $|G|$. Then up to conjugacy in $G$ the following assertions hold:

(a) $r \neq 2$ and $R$ is a Sylow $r$-subgroup of $G$. In addition $C$ is a maximal torus $T_i$ of $G$ and $N/C \simeq \mathbb{Z}_2$ or $\mathbb{Z}_6$ according as $i = 1$ or $i \in \{2, 3, 4\}$.

(b) $r = 2$ and $R \in \{\langle u \rangle, \langle u, u' \rangle, S\}$, where $S$ is a Sylow 2-subgroup of $G$. In addition, if $|R| = 2$, then $N = C = \mathbb{Z}_2 \times L_2(q^2)$. If $|R| = 4$, then $C = R \times D_{\frac{q}{2} + 1}(q^2 + 1)$ and $N = C \times \mathbb{Z}_3$. If $R = S$, then $C = R$ and $N/C \simeq \mathbb{Z}_7 \times \mathbb{Z}_3$.

2. The Blocks

The notation and terminology of Section 1 are continued in this section. The blocks and the number of irreducible Brauer characters of a block of $^2G_2(q^2)$ are given in this section.

Let $R$ be an $r$-subgroup of a finite group $G$, $C = C_G(R)$, and $N = N_G(R)$. Let $B$ be an $r$-block of $G$. If $b$ is an $r$-block of $C$, then $(R, b)$ is called a Brauer ($r$-)pair of $G$ (by Alperin/Broué), and is a $B$-pair if $B = bG$. Thus $(R, b)$ is a $B$-pair if and only if $(1, B) \leq (R, b)$. In addition pairs $(1, B)$ correspond to blocks $B$ of $G$. We denote by $\text{Irr}(G)$ and $\text{Irr}(B)$ the sets of irreducible characters of $G$ and $B$, respectively.

Let $G = ^2G_2(q^2)$ and let $G^*$ be its dual group. Then $G$ and $G^*$ are isomorphic (cf. [10], p. 120), and we shall identify them. Let $s$ be a semisimple element of $G$, and let $(s)$ be the conjugacy class of $s$ in $G$. Following [9], p.57, we denote by $\mathcal{E}(G, (s))$ the set of the irreducible constituents of Deligne-Lusztig generalized characters associated with $(s)$ (cf. [9], p. 57). Moreover given a semisimple $r'$-element $s$ of $G$, let

$$\mathcal{E}_r(G, (s)) = \bigcup_y \mathcal{E}(G, (sy)),$$

where $y$ runs over all the $r'$-elements of $C_G(s)$. By Theorem 2.2 of [9], $\mathcal{E}_r(G, (s))$ is a union of $r$-blocks. An irreducible character in $\mathcal{E}(G, (1))$ is an irreducible unipotent character. Thus $G$ has 8 irreducible unipotent characters, the trivial character
\( \xi_1 = 1 \), the Steinberg character \( \xi_3 = St \), and 6 cuspidal unipotent characters \( \xi_5 - \xi_{10} \) in the notation of \([22]\) (see \([10]\), pp. 488, 489). We say that a character \( \chi \in \text{Irr}(G) \) is of \( r \)-defect 0 if the \( r \)-part \( \chi(1)_r \) of the degree \( \chi(1) \) equals the \( r \)-part \( \#G \_r \) of \( \#G \). If \( \chi \) is of defect 0, then the block \( B(\chi) \) of \( G \) containing \( \chi \) has the trivial defect group and \( |\text{Irr}(B)| = 1 \). Given a semisimple \( r' \)-element \( s \) of \( G \), let \( \mathcal{E}_r(G, (s))' \) be the subset of characters in \( \mathcal{E}_r(G, (s)) \) with a non-zero \( r \)-defect. Thus \( \mathcal{E}_r(G, (s))' \) is also a union of \( r \)-blocks by \([9]\), Theorem 2.2.

**Proposition (2A).** Let \( G = 2G_2(q^2) \) and let \( B \) be an \( r \)-block of \( G \) with a non-trivial defect group \( D \). Suppose that \( B \subseteq \mathcal{E}_r(G, (s)) \) for some semisimple \( r' \)-element \( s \) of \( G \).

(a) Suppose that \( r \neq 2 \). Then \( B = \mathcal{E}_r(G, (s))' \) and \( D \) is a Sylow \( r \)-subgroup of \( G \).

(b) Suppose that \( r = 2 \). Then \( B = \mathcal{E}_r(G, (s))' \) and \( D \in \{ \langle u \rangle, \langle u, u' \rangle, S \} \).

(c) If \( \ell(B) \) is the number of irreducible Brauer characters in \( B \), then

\[
\ell(B) = \begin{cases} 
2 & \text{if } r \neq 2 \text{ and } |s| = 2, \\
2 & \text{if } r \text{ divides } \frac{1}{2}(q^2 - 1) \text{ and } B \text{ is the principal block,} \\
6 & \text{if } r \neq 2, r \text{ does not divide } \frac{1}{2}(q^2 - 1), \text{ and } B \text{ is the principal block,} \\
5 & \text{if } r = 2 \text{ and } B \text{ is the principal block,} \\
1 & \text{otherwise.}
\end{cases}
\]

**Proof.** (1) Suppose that \( r \neq 2 \). By Proposition 3.2 of \([16]\) \( B = \mathcal{E}_r(G, (s))' \) and since \( r \neq 3, r \) is a good prime (cf. \([10]\), p. 183). By Theorem 5.1 of \([14]\) the restrictions of characters in \( \mathcal{E}(G, (s)) \) to the \( r \)-regular elements form a basic set of Brauer characters in \( \mathcal{E}_r(G, (s)) \). Since irreducible Brauer characters are linear independent and the restriction of each character in \( \text{Irr}(B) \) to the \( r \)-regular elements is a linear combination of irreducible Brauer characters in \( B \), it follows that \( \ell(B) = |\mathcal{E}(G, (s))'| \). If \( s = 1 \), then \( B = \mathcal{E}_r(G, (1))' \) is the principal block and by the degrees of unipotent characters given by Lemma 3.1 of \([16]\), \( |\mathcal{E}(G, (1))'| = 2 \) or 6 according as \( r \) divides or does not divide \( q^2 - 1 \). Thus (c) holds. If \( s \neq 1 \), then (c) follows by \( |\mathcal{E}(G, (s))| = |\mathcal{E}(C_G(s), (1))| \).

(2) Suppose that \( r = 2 \). By \([22]\) or \([19]\), p. 83 the principal block \( B_1 \) has 8 irreducible characters and 5 irreducible Brauer characters. If \( \xi \in \mathcal{E}_2(G, (t)) \), then \( \xi(1) = |G: C_G(t)| \cdot \lambda(1) \), where \( \lambda \) is an irreducible unipotent character of \( C_G(t) \). But \( \mathcal{E}_2(G, (1)) = \mathcal{E}(G, (1)) \cup \mathcal{E}(G, (u)) \), so by the degrees of irreducible characters in \([22]\), \( \mathcal{E}_2(G, (1)) = \{ \xi_i : 1 \leq i \leq 10 \} \), where \( \xi_i \) are given by \([22]\), p. 87. Since \( \mathcal{E}_2(G, (1)) \) has two irreducible unipotent characters, \( \xi_9 \) and \( \xi_{10} \) of defect 0, it follows that \( B_1 = \mathcal{E}_2(G, (1))' \).

Suppose that a defect group \( D \) of \( B \) has order 2 or 4. Then we may suppose that \( D = \langle u \rangle \) or \( \langle u, u' \rangle \), so that \( C_G(D) = D \times L_2(q^2) \) or \( D \times D_{\frac{1}{2}}(q^2+1) \), respectively. By the Extended First Main Theorem of Brauer \([13]\), Theorem III.9.7 and Lemmas V.3.6, V.3.7, \( B = bG \) for some block \( b \) of \( C_G(D) \) with a defect group \( D \). If \( b \subseteq \)
\( \mathcal{E}_2(C_G(D), (t)) \) for some semisimple 2'-element \( t \) of \( C_G(D) \), then by [9], Theorem 3.2, \( t \) and \( s \) are conjugate in \( G \). We may suppose that \( s \in C_G(D) \), so that \( D \leq C_G(s) \). If \( |D| = 4 \), then by (1.2), we may suppose that \( s \in T_2 \) and so \( C_G(s) = T_2 \). Moreover \( C_G(sy) = T_2 \) for each \( y \in D = O_2(T_2) \). But a Deligne-Lusztig generalized character \( R_{T_2}(sy) \) has degree \( (q^2 - 1)(q^4 - q^2 + 1) \) (see [10], Theorem 7.5.1) and by [22], p. 75, \( B \) has 4 irreducible characters of degree \((q^2-1)(q^4-q^2+1)\), so each \( \pm R_{T_2}(sy) \) is irreducible for a suitable sign \( \pm \). It follows that \( |\mathcal{E}_2(G, (s))| = 4 \) and \( B = \mathcal{E}_2(G, (s)) \). Suppose that \( |D| = 2 \). If \( s \in T_2 \), then each block contained in \( \mathcal{E}_2(G, (s)) \) has a defect group of order 4 as shown above. This contradiction implies that \( s \in T_1 \). A proof similar to above shows that \( B = \mathcal{E}_2(G, (s)) \) and \( |\text{Irr}(B)| = 2 \).

Finally we count \( \ell(B) \) for a non-principal block \( B \). Given semisimple 2'-element \( t \) of \( G \), let \( B_t \) be the block \( \mathcal{E}_2(G, (t))' \). Thus \( \ell(B_t) \geq 1 \) and \( \ell(B_1) = 5 \). By the character table of \( G \), [22], pp. 87–88, \( G \) has two blocks, denoted by \( B_1' \) and \( B_1'' \), of defect 0, and both are contained in \( \mathcal{E}_2(G, (1)) \). Thus \( \ell(B_1') = \ell(B_1'') = 1 \). By a result of Brauer the number of irreducible Brauer characters in \( G \) is equal to the number of conjugacy 2'-classes of \( G \). Thus

\[
2 + 5 + \sum_{t \neq 1} \ell(B_t),
\tag{2.1}
\]

is the number of 2'-classes of \( G \), where \( t \) runs over the representatives for the non-trivial semisimple conjugacy 2'-classes of \( G \). On the other hand two elements \( g \) and \( g' \) are conjugate in \( G \) if and only if both the semisimple parts \( g_s, g'_s \) and the unipotent parts \( g_u, g'_u \) of \( g \) and \( g' \) are conjugate in \( G \), respectively. Thus we may suppose that \( g_s = g'_s \), so \( g_u \) and \( g'_u \) are conjugate in \( C_G(g_s) \). For a semisimple 2'-element \( t \) of \( G \), let \( u(t) \) be the number of unipotent conjugacy classes of \( C_G(t) \). By Chapter III of [22] \( G \) has 7 unipotent conjugacy classes, so that \( u(1) = 7 \). Thus the number of conjugacy 2'-classes of \( G \) is

\[
7 + \sum_{t \neq 1} u(t),
\tag{2.2}
\]

where \( t \) runs over the representatives for the non-trivial semisimple conjugacy 2'-classes of \( G \). If \( t \neq 1 \), then \( C_G(t) \) is a maximal torus and so \( u(t) = 1 \). It follows by (2.1) and (2.2) that \( \ell(B_t) = 1 \) for \( t \neq 1 \). Thus (c) follows.

**Remark.** From (2A), Brauer's height conjecture and the Alperin-McKay conjecture for the group \( G = ^2G_2(q^2) \) can be verified easily. The truth of these two conjectures was also pointed out by G.O. Michler [20].

### 3. The Weights

The notation and terminology of Sections 1 and 2 are continued in this section. The numbers of weights of blocks of \( ^2G_2(q^2) \) are given in this section.

Given a weight \((R, \varphi)\) of a finite group \( G \), let \( C = C_G(R), N = N_G(R), \) and let \( \theta \) be an irreducible constituent of the restriction \( \varphi|_{CR} \) of \( \varphi \) to \( CR \). By Clifford theory \( \theta \) acts trivial on \( R \) and has defect 0 as a character of \( CR/R \). Let \( N(\theta) \) be the stabilizer of \( \theta \) in \( N \). Denote by \( \text{Irr}^0(N(\theta), \theta) \) and \( \text{Irr}^0(N, \theta) \) the sets of irreducible
characters of $N(\theta)$ and $N$ which cover $\theta$ and which have defect 0 as characters of $N(\theta)/R$ and $N/R$, respectively. Then $\varphi \in \text{Irr}^0(N,\theta)$ since $\varphi$ has defect 0 as a character of $N/R$. By Clifford theory, the induction mapping $\psi \mapsto I(\psi) = \text{Ind}_{N(\theta)}^N(\psi)$ induces a bijection from $\text{Irr}^0(N,\theta)$ to $\text{Irr}^0(N,\theta)$.

The block $b$ of $CR$ containing $\theta$ has a defect group $D$ by [13], Lemma 4.4. By [2], p. 3 all $B$-weights for a block $B$ of $G$ have the form $(R, I(\psi))$, where $R$ runs over representatives for the conjugacy $G$-classes of radical subgroups, $b$ runs over representatives for the conjugacy $N$-classes of blocks of $CR$ such that $b$ has defect group $R$ and $b^G = B$, and $\psi$ runs over $\text{Irr}^0(N(\theta),\theta)$. Here $\theta$ is the canonical character of $b$. If $b^G = B$ and $R$ is a defect group of $b$, then by [13], Lemma V.6.1,

$$Z(D) \leq Z(R) \leq R \leq D$$

for some defect group $D$ of $B$. In particular, $R = D$ whenever $D$ is abelian.

Let $G = 2G_2(q^2)$ and let $B$ be a block of $G$ with a defect group $D$. Suppose that $B \subseteq \mathcal{E}_2(G, (s))$ for some semisimple $r'$-element $s$ of $G$ and suppose that $(R, \varphi)$ is a $B$-weight. Then we may suppose that $R = D$ and $\varphi \in \text{Irr}^0(N,\theta)$, where $N = N_G(D)$ and $\theta$ is an irreducible constituent of the restriction $\varphi|_{C_G(D)}$ of $\varphi$ to $C_G(D)$. Let $C = C_G(D)$ and let $b$ be the block of $C$ containing $\theta$. Then $b^G = B$ and $b$ has a defect group $D$. By the Extended First Main Theorem of Brauer $C$ has only one $N$-class of blocks $b'$ such that $b'^G = B$ as $D$ is a defect group of $B$. Thus the number of $B$-weights is equal to $|\text{Irr}^0(N(\theta),\theta)|$.

If $B$ is the principal block, then so is $b$ by Brauer's Third Main Theorem, [13], Theorem V.6.2. Thus the canonical character $\theta$ of $b$ is the trivial character of $C$. So $N(\theta) = N_G(D)$, and $\text{Irr}^0(N(\theta),\theta) = \text{Irr}(N(\theta)/C_G(D))$ as, by (1.4) $r$ and $|N_G(D)/C_G(D)|$ are relatively prime. If $C_G(s)$ is a maximal torus $T$, then $D = O_r(T)$. As shown in the proof of (2A) (2) $\pm R_T(sy)$ is irreducible for a suitable sign $\pm$ when $r = 2$. If $r \neq 2$, then a similar proof to that of (2A) (2) shows that $\pm R_T(sy)$ is also irreducible by using the degrees of characters in $\text{Irr}(B)$ and the character table of [22]. By Theorem 7.3.4 of [10] the character $\phi(s)$ in $\text{Irr}(T)$ is in the general position, in the sense of [10], p. 219, where $\phi(s)$ corresponds to $s$ under the correspondence of Deligne-Lusztig. It follows by [10], Theorem 7.3.4 again that $N(\theta) = T$, so that $|\text{Irr}^0(N(\theta),\theta)| = 1$. In particular $N_{C_G(s)}(D)/C_G(D) = 1$.

Finally, suppose that $C_G(s)$ is not a maximal torus and $s \neq 1$. Then $r \neq 2$ and we may suppose that $s = u$. Thus $C_G(D)$ is a maximal torus $T$, $N_G(D) = N_G(T)$, and $C_G(s) = \langle s \rangle \times L_2(q^2)$. So $T = T_1$ or $T_2$. By [9], Theorem 3.2 we may suppose that $s \in T$ and $b = \mathcal{E}_r(T, (s))$. Let $\phi$ be an isomorphism from $T$ to $\text{Irr}(T)$ such that $\phi$ induces the bijection between the conjugacy $G$-classes of $(T,t)$ and the $G$-classes of $(T,\phi(t))$, where $t \in T$. Thus $\theta = \phi(s)$ as $b = \mathcal{E}_r(T, (s))$. If $\tau$ is an involution of $N_G(T)/T$, then $\tau$ centralizes each element of $O_2(T)$ as a Sylow 2-subgroup of $G$ is abelian. Thus $\tau$ stabilizes each character in $\text{Irr}(T)$ of order 2, so that $\tau \in N(\theta)$. If $T = T_1$, then $N(\theta) = N_G(D)$. If $T = T_2$, then an element $\sigma$ in $N_G(T)/T$ of order 3 permutes the three involutions of $O_2(T)$. Thus $\sigma$ acts non-trivially on the three involutions of $\text{Irr}(T)$, so that $\sigma \notin N(\theta)$. In both cases $N(\theta)/T \cong \mathbb{Z}_2$. By Clifford theory $|\text{Irr}^0(N(\theta),\theta)| = 2$. Since $N_{C_G(s)}(D)/C_G(D) \cong \mathbb{Z}_2$, it follows that $|\text{Irr}^0(N(\theta),\theta)| = |\text{Irr}(N_{C_G(s)}(D)/C_G(D))|$. Thus we have proved the first part of the following theorem.
**Theorem (3A).** Let $G = 2G_2(q^2)$ and let $B$ be an $r$-block of $G$ with a non-trivial defect group $D$. Suppose that $B \subseteq \mathcal{E}_r(G,(s))$ for some semisimple $2'$-element $s$ of $G$. Then we may suppose that $s \in C_G(D)$, so that $D \leq C_G(s)$.

(a) If $W(B)$ is the number of $B$-weights, then

$$W(B) = |\text{Irr}(N_{C_G(s)}(D)/C_G(D))|.$$ 

(b) $W(B)$ is the number $\ell(B)$ of irreducible Brauer characters in $B$.

**Proof.** (b) If $C_G(s)$ is a maximal torus, then by (2A) (c), $\ell(B) = 1$, so that $\ell(B) = W(B)$. If $r \neq 2$ and $|s| = 2$, then by (2A) (c), $\ell(B) = 2$, so that $\ell(B) = W(B)$. Thus we may suppose that $B$ is the principal block, so that $C_G(s) = G$ and $N_{C_G(s)}(D) = N_G(D)$. If $r \neq 2$, then $C_G(D)$ is a maximal torus $T$ and $N_G(D) = N_G(T)$. By (1.4) $N_G(T)/T \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_6$ according as $T = T_1$ or $T \neq T_1$. In the former case $r$ divides $q^2 - 1$ and in the latter case $r$ does not divide $q^2 - 1$. Thus (b) follows by (a) and (2A) (c).

Finally suppose that $r = 2$. By [19], p. 84 $N_G(D)/C_G(D)$ is the Frobenius group $F_{21}$ of order 21. Thus the Sylow 7-subgroup $K$ of $F_{21}$ is the Frobenius kernel and a Sylow 3-subgroup $H$ is a Frobenius complement (cf. [15], p. 38). So $F_{21} = K \rtimes H$. By [15], Theorem 4.5.3 a generator of $H$ stabilizes only the trivial character of $K$, so by Clifford theory, $F_{21}$ has three linear characters and two irreducible characters of degree 3. It follows that $|\text{Irr}(F_{21})| = 5$ and (b) follows by (2A) (c). This completes the proof.

**References**


17. P. Kleidman, *The maximal subgroups of the Chevalley groups G_2(q) with q odd, the Ree groups 2G_2(q) and their automorphism groups*, J. Algebra **117** (1988), 30–71.


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