A SHORT PROOF OF THE WEDDERBURN-ARTIN THEOREM

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(Received December 1991)

Abstract. The Wedderburn-Artin theorem is of fundamental importance in noncommutative ring theory. A short self-contained proof is given which requires only elementary facts about rings.

Throughout this note $R$ will denote an associative ring with unity $1 \neq 0$. If $X$ and $Y$ are additive subgroups of $R$, define their product by

$$XY = \left\{ \sum_{i=1}^{n} x_i y_i \mid n \geq 1, \ x_i \in X, \ y_i \in Y \right\}.$$  

This is an associative operation. An additive subgroup $K$ is called a left (right) ideal of $R$ if $RK \subseteq K$, and $K$ is called an ideal if it is both a left and right ideal. The ring $R$ is called semiprime if $A^2 \neq 0$ for every nonzero ideal $A$, and $R$ is called left artinian if it satisfies the descending chain condition on left ideals (equivalently, every nonempty family of left ideals has a minimal member).

The following theorem is a landmark in the theory of noncommutative rings.

**Wedderburn-Artin Theorem.** If $R$ is a semiprime left artinian ring then

$$R \cong M_{n_1}(D_1) \times M_{n_2}(D_2) \times \ldots \times M_{n_r}(D_r)$$

where each $D_i$ is a division ring and $M_n(D)$ denotes the ring of $n \times n$ matrices over $D$.

In this form the theorem was proved [1] in 1927 by Emil Artin (1898–1962) generalizing the original 1908 result [4] of Joseph Henry Maclagan Wedderburn (1882–1948) who proved it for finitely generated algebras over a field. The purpose of this note is to give a quick, self-contained proof of this theorem. A key result is the following observation [2] of Richard Brauer (1902–1977). Call a left ideal $K$ minimal if $K \neq 0$ and the only left ideals contained in $K$ are $0$ and $K$.

**Brauer's Lemma.** Let $K$ be a minimal left ideal of a ring $R$ and assume $K^2 \neq 0$. Then $K = Re$ where $e^2 = e \in R$ and $eRe$ is a division ring.

**Proof.** Since $0 \neq K^2$, certainly $Ku \neq 0$ for some $u \in K$. Hence $Ku = K$ by minimality, so $eu = u$ for some $e \in K$. If $r \in K$, this implies $re-r \in L = \{a \in K \mid au = 0\}$. Now $L$ is a left ideal, $L \subseteq K$, and $L \neq K$ because $eu \neq 0$. So $L = 0$ and it follows that $e^2 = e$ and $K = Re$.

Now let $0 \neq b \in eRe$. Then $0 \neq Rb \subseteq Re$ so $Rb = Re$ by minimality, say $e = rb$. Hence $(ere)b = er(eb) = erb = e^2 = e$, so $b$ has a left inverse in $eRe$. It follows that $eRe$ is a division ring.

The following consequence will be needed later.

1991 AMS Mathematics Subject Classification: 16P20
Corollary. Every nonzero left ideal in a semiprime, left artinian ring contains a nonzero idempotent.

Proof. If $L \neq 0$ is a left ideal of $R$, the left artinian condition gives a minimal left ideal $K \subseteq L$. Now $(KR)^2 \neq 0$ because $R$ is semiprime, so $(KR)^2 = KRKR \subseteq K^2R$ shows that $K^2 \neq 0$. Hence Brauer’s lemma applies.

A ring $R$ is simple if $R$ has no ideals other than 0 and $R$. Such a ring is necessarily semiprime. When $R$ is simple the Wedderburn-Artin theorem is known as Wedderburn’s Theorem and a short proof is well known (see Henderson [3]). Since this result is needed in the general case, we sketch the proof. The left artinian hypothesis is weakened to the existence of a minimal left ideal.

Wedderburn’s Theorem. If $R$ is a simple ring with a minimal left ideal, then $R \cong M_n(D)$ for some $n \geq 1$ and some division ring $D$.

Proof (Henderson). Let $K$ be a minimal left ideal. Then $KR = R$ (it is a nonzero ideal) so $R = R^2 = (KR)^2 = KRKR \subseteq K^2R$. Hence $K^2 \neq 0$ so, by Brauer’s lemma, $K = Re$ where $e^2 = e$ and $D = eRe$ is a division ring. Then $K$ is a right vector space over $D$ and, if $r \in R$, the map $\alpha_r : K \to K$ given by $\alpha_r(k) = rk$ is a $D$-linear transformation. Hence $r \to \alpha_r$ is a ring homomorphism $R \to \text{end}_D K$, and it is one-to-one because $\alpha_r = 0$ implies $rRe = 0$ so $0 = rRRe = rR (ReR = R$ because $R$ is simple). To see that it is onto, write $1 \in ReR$ as $1 = \sum_{i=1}^n r_i es_i$. Given $\alpha \in \text{end}_D K$, let $a = \sum_i \alpha(r_i e) es_i$. Then the $D$-linearity of $\alpha$ gives

$$\alpha(re) = \alpha \left[ \sum_i (r_i es_i)re \right] = \sum_i \alpha(r_i e)(es_i re) = a \cdot re = \alpha_a(re)$$

for all $r \in R$, so $\alpha = \alpha_a$. Thus $R \cong \text{end}_D K$ and it remains to show that $K_D$ is finite dimensional (then $\text{end}_D K \cong M_n(D)$ where $n = \dim_D K$). But if $\dim_D K$ is infinite, the set $A = \{ \alpha \in \text{end}_D K \mid \alpha(K)$ has finite dimension} is a proper ideal of $\text{end}_D K$, contrary to the simplicity of $R$.

It is worth noting that, if $e^2 = e \in R$ is such that $ReR = R$, the proof shows that $R \cong \text{end}_D K$ where $K = Re$ is regarded as a right module over $D = eRe$.

To prove the Wedderburn-Artin theorem, it is convenient to introduce a weak finiteness condition in a ring $R$. Let $I$ denote the set of idempotents in $R$. Given $e, f$ in $I$, write $e \leq f$ if $ef = e = fe$, that is if $eRe \subseteq fRf$. This is a partial ordering on $I$ (with 0 and 1 as the least and greatest elements) and $I$ is said to satisfy the maximum condition if every nonempty subset contains a maximal element, equivalently if $e_1 \leq e_2 \leq \ldots$ in $I$ implies $e_n = e_{n+1} = \ldots$ for some $n \geq 1$. The minimum condition on $I$ is defined analogously. A set of idempotents is called orthogonal if $ef = 0$ for all $e \neq f$ in the set.

Lemma 1. The following are equivalent for a ring $R$:
(1) \( R \) has maximum condition on idempotents.

(2) \( R \) has minimum condition on idempotents.

(3) \( R \) has maximum condition on left ideals \( Re, \ e^2 = e \) (on right ideals \( eR, \ e^2 = e \)).

(4) \( R \) has minimum condition on left ideals \( Re, \ e^2 = e \) (on right ideals \( eR, \ e^2 = e \)).

(5) \( R \) contains no infinite orthogonal set of idempotents.

**Proof.** The verification that (1) \( \Leftrightarrow \) (2), (3) \( \Leftrightarrow \) (4) and (3) \( \Rightarrow \) (5) \( \Rightarrow \) (1) are routine, so we prove that (1) \( \Rightarrow \) (3).

If \( Re \subseteq Re_2 \subseteq \ldots \) where \( e_i^2 = e_i \) for each \( i \), then \( e_i e_j = e_i \) for all \( j \geq i \) so we inductively construct idempotents \( f_1 \leq f_2 \leq \ldots \) as follows: Take \( f_1 = e_1 \) and, if \( f_i \) has been specified, take \( f_{i+1} = f_i + e_{i+1} - e_{i+1} f_i \). An induction shows that \( f_i \in Re_i \) for each \( i \), whence \( f_i e_k = f_i \) for all \( k \geq i \). Using this one verifies that \( f_i^2 = f_i \) and \( f_i \leq f_{i+1} \) hold for each \( i \geq 1 \). Thus (1) implies that \( f_n = f_{n+1} = \ldots \) for some \( n \) and hence that \( e_{i+1} = e_{i+1} f_i \in Re_i \) for all \( i \geq n \). It follows that \( R e_n = R e_{n+1} = \ldots \). The maximum condition on right ideals \( eR \) is proved similarly.

Call a ring \( R \) *I-finite* if it satisfies the conditions in Lemma 1. It is clear that every left (or right) artinian or noetherian ring is I-finite.

**Proof of the Wedderburn-Artin Theorem.** Let \( R \) be a semiprime, left artinian ring, let \( K \) be a minimal left ideal, let \( S = Kr \), and let \( M = \{a \in R \mid Sa = 0\} \). Then \( S \) and \( M \) are ideals of \( R \) and we claim that

\[
R = S \oplus M. \tag{*}
\]

First \( S \cap M = 0 \) because \( R \) is semiprime and \( (S \cap M)^2 \subseteq SM = 0 \). Since \( R \) is I-finite, let \( e \) be a maximal idempotent in \( S \). To show that \( R = S + M \), it suffices to show \( 1 - e \in M \). If not, then \( S(1-e) \neq 0 \) so (by the Corollary to Brauer's lemma) let \( f \in S(1-e) \) be a nonzero idempotent. Then \( fe = 0 \) and one verifies that \( g = e + f - ef \) is an idempotent in \( S \) and \( e \leq g \). The maximality of \( e \) then gives \( e = g \), so \( f = ef \), whence \( f = f^2 = fef = 0 \), a contradiction. So \( 1 - e \in M \) and \( R = S + M \), proving (\( * \)).

Hence \( S \) and \( M \) are rings (with unity) and they inherit the hypotheses on \( R \) because left ideals of \( S \) or \( M \) are left ideals of \( R \) by (\( * \)). Moreover, this shows that \( S \) is simple. Indeed, if \( A \neq 0 \) is an ideal of \( S \) then \( A \cap K \neq 0 \) (otherwise \( A^2 \subseteq AKR \subseteq (A \cap K)R = 0 \)) so the minimality of \( K \) gives \( K \subseteq A \), whence \( S = Kr \subseteq A \).

If \( M = 0 \) the proof is complete by Wedderburn's theorem. Otherwise, repeat the above with \( R \) replaced by \( M \) to get \( R = S \oplus S_1 \oplus M_1 \) where \( S_1 \) is simple. This cannot continue indefinitely by the artinian hypothesis (or I-finiteness), so Wedderburn's theorem completes the proof.

**Remark 1.** The converse to both these theorems is true.

**Remark 2.** These proofs actually yield the following: A ring \( R \) is semiprime and left artinian if and only if it satisfies the following condition.

\[
R \text{ is I-finite and every nonzero left ideal contains a nonzero idempotent.} \tag{**}
\]
The necessity of (**) follows from Lemma 1 and the Corollary to Brauer's lemma. Conversely, if $R$ satisfies (**) then the proofs of both theorems go through virtually as written once the following is established: If $E$ is a minimal nonzero idempotent, then $Re$ is a minimal left ideal. But if $L \subseteq Re$ is a left ideal and $L \neq 0$, let $0 \neq f^2 = f \in L$. Then $fe = f$ so $g = ef \in L$ is an idempotent, $g \neq 0$ (because $f = fg$) and $g \leq e$. Thus $g = e$ by the minimality of $e$, whence $L = Re$.

References