A BANACH LATTICE CHARACTERIZATION
OF $c_0$ AND $\ell_p$

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1. Introduction

We show that a 3-dimensional Banach lattice has $p$-additive norm if any sublattice through either of two given axes is the range of a contractive projection. A Banach lattice is $c_0$ or $\ell_p$ for some $1 \leq p < \infty$ if and only if there is a sequence $(e_n)$ of positive, mutually disjoint vectors whose span is dense such that every 2-dimensional sublattice containing one of them is the range of a contractive projection. We characterize $\ell_p$ direct products of Banach lattices with ideals having $p$-additive norm and give new conditions that imply that a Banach lattice has $p$-additive norm.

Ando characterized Banach lattices which have $p$-additive norm in [1]. He looked at a three-dimensional sublattice and assumed that sublattices through the co-ordinate axes were the ranges of contractive projections (that is, projections of norm 1) to show that the norm had to be $p$-additive.

In this paper we show that it is enough to assume that sublattices through two of the axes of a three dimensional sublattice (we take the first and third) are the range of contractive projections. In section 2 the function $F(t) = \|e_1 + te_2\|$ is shown to satisfy the functional equation $F(st/F(s)) = F(s)F(st/F(s))$. In section 3 we solve that functional equation to show that the solutions are $F(t) = \|(1,t)\|_p$ where $1 \leq p \leq \infty$, which implies that $\|x_1e_1 + x_2e_2 + x_3e_3\| = \|(x_1,x_2,x_3)\|_p$.

In Section 4 we apply our result to characterize the Banach lattices $c_0$ and $\ell_p$ ($1 < p < \infty$) as those Banach lattices $X$ which have a countable disjoint sequence $(e_n)$ of positive vectors such that the span of the vectors $e_n$ is dense in $X$ and any two dimensional sublattice which contains one of the $e_n$ is the range of a contractive projection on $X$. We also show that a Banach lattice $X$ is the $\ell_p$ sum of ideals $A$ and $B$ with $A$ having $p$-additive norm provided any two dimensional sublattice of $X$ which

intersects $A \setminus \{0\}$ is the range of a contractive projection. By reducing the class of two-dimensional sublattices which are assumed to be the ranges of contractive projections we get new conditions which imply that a Banach lattice has $p$-additive norm.

Definition 1.1. For $1 \leq p \leq \infty$ the norm on a Banach lattice $X$ is $p$-additive provided that for vectors $x$ and $y$ with $x \wedge y = 0$ we have

$$\|x + y\|^p = \|x\|^p + \|y\|^p \quad (p < \infty) \text{ or }$$

$$\|x + y\| = \max(\|x\|, \|y\|) \quad (p = \infty).$$

If the norm on $X$ is $p$-additive then $X$ is called an abstract $L_p$ space if $p < \infty$ and an abstract $M$-space if $p = \infty$.

We refer the reader to Lacey's book [4, §15, Theorem 3] for the following characterization, due to Bohnenblust and others, of complex abstract $L_p$ spaces.

Theorem 1.2. A complex Banach lattice whose norm is $p$-additive, $1 \leq p < \infty$, is linearly isometric and lattice isomorphic to $L_p(\mu, C)$ for some measure $\mu$.

For abstract $M$-spaces the situation is too complicated to be summarized here: the reader is again referred to Lacey's book [4, §9].

2. Three dimensional Banach Lattices

Suppose $X$ is a 3-dimensional Banach lattice with unit vectors $e_1$, $e_2$ and $e_3$ such that $e_i \wedge e_j = 0$ if $i \neq j$. $X$ may be over the real or complex numbers. We extract the following result from the first half of the proof of Theorem 4 of §16 of Lacey's book [4, pp.143-144]; the idea is due to Ando [1]. It appears also in [5, Lemma 1.11].

Lemma 2.1. If every 2-dimensional vector sublattice of $X$ which contains $e_1$ is the range of a positive contractive projection on $X$ then there is a function $H : [0,1) \rightarrow (0,1]$ such that if $0 \leq \alpha < 1$, $\beta \geq 0$ and $\gamma \geq 0$ then $\|\alpha e_1 + \beta e_2 + \gamma e_3\| = 1$ if and only if $\|\beta e_2 + \gamma e_3\| = H(\alpha)$.

We use this result to prove the following equality between norms of certain elements of $X$.
intersects \( A \setminus \{0\} \) is the range of a contractive projection. By reducing the class of two-dimensional sublattices which are assumed to be the ranges of contractive projections we get new conditions which imply that a Banach lattice has \( p \)-additive norm.

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\[
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\]

If the norm on \( X \) is \( p \)-additive then \( X \) is called an abstract \( L_p \) space if \( p < \infty \) and an abstract \( M \)-space if \( p = \infty \).

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**Theorem 1.2.** A complex Banach lattice whose norm is \( p \)-additive, \( 1 \leq p < \infty \), is linearly isometric and lattice isomorphic to \( L_p(\mu, \mathbb{C}) \) for some measure \( \mu \).

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Suppose \( X \) is a 3-dimensional Banach lattice with unit vectors \( e_1 \), \( e_2 \) and \( e_3 \) such that \( e_i \wedge e_j = 0 \) if \( i \neq j \). \( X \) may be over the real or complex numbers. We extract the following result from the first half of the proof of Theorem 4 of §16 of Lacey's book [4, pp.143-144]; the idea is due to Ando [1]. It appears also in [5, Lemma 1.b.11].

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We use this result to prove the following equality between norms of certain elements of \( X \).
Suppose $H$ contains $e_1$ say, $H = \{x, e_i : a_1 x_1 = a_3 x_3\}$ for $a_i \geq 0$.

Let the given contractive projection $P$ be along $\langle x \rangle$. By symmetry of the norm, if $x$ and $x'$ are symmetric with respect to the $e_2e_3$ plane, the projection along $\langle x' \rangle$ is contractive. By the convexity result, the projection $R$ along $\langle x + x' \rangle$ is contractive.

To show $R$ is positive we only have to show its restriction to the $e_2e_3$ plane is positive, i.e. it takes points in the 1st quadrant to points in the 1st quadrant.

The line in the $\langle e_2 \rangle$ direction is tangent at $e_3$ to the unit ball, and the line in the $\langle e_3 \rangle$ direction is tangent at $e_2$. Take $y$ in the $e_2e_3$ plane, in $H$, with $\|y\| = 1$, $y$ in the 1st quadrant. Any tangent at $y$ must be in a direction in the 2nd and 4th quadrants, and hence $\langle x + x' \rangle$ is in the 2nd and 4th quadrants. Hence the projection along $\langle x + x' \rangle$ takes points in the 1st quadrant to points in the 1st quadrant.

Remark. In the above result, not every contraction need be positive. For example, with $L_1$ norm $\|x\| = |x_1| + |x_2| + |x_3|$ we can project onto the plane $x_2 = 0$ along any line $\langle y_1 e_1 + y_2 e_2 + y_3 e_3 \rangle$ with $y_1$ and $y_3$ small and have a contraction. But it is not positive if one of $y_1$ or $y_3$ are $> 0$.

Proposition 2.5. Suppose every two dimensional vector sublattice which contains $e_1$ or $e_3$ is the range of a contractive projection on $X$. For

\[ x = x_1 e_1 + x_2 e_2 + x_3 e_3, \quad x_i > 0, \]
\[ \|x\| = x_1 F(x_2^{-1} F(x_3^{-1} x_1)) \]
\[ = x_1 F(x_2^{-1} F(x_3^{-1} / F(x_2^{-1} x_1))). \]

For $s, t > 0$,

\[ F(sF(t)) = F(s)F(st/F(s)). \quad (2.1) \]

Proof. Let $x_1, x_2, x_3 > 0$. By Proposition 2.2, with $e_1$ and $e_3$ interchanged,
\[ \|x_1 e_1 + x_2 e_2 + x_3 e_3\| = \|x_3 e_3 + x_1 e_1 + x_2 e_2\| = \|x_3 e_3 + x_1 e_1 + x_2 e_2 e_2\| .\]

Thus \( \|se_1 + te_2\| = \|se_1 + te_3\| = \|se_2 + te_3\| \) for \( s, t \geq 0 \).

Hence we have

\[ \|x_1 e_1 + x_2 e_2 + x_3 e_3\| = \|x_1 e_1 + x_2 e_2 + x_3 e_2\| = \|x_1 e_1 + x_2 e_2 + x_3 e_2 e_2\| = \|x_3 e_3 + x_1 e_1 + x_2 e_2\| = \|x_3 e_2 + x_1 e_1 + x_2 e_2 e_2\| .\]

Since \( \|x_1 e_1 + x_2 e_1 + x_3 e_2\|_2 = x_1 - F(x_2 x_2^{-1} F(x_3 x_3^{-1})) \) and

\[ \|x_2 e_2 + (x_1 e_1 + x_2 e_2)\|_1 = x_1 - F(x_2 x_2^{-1}) F(x_3 x_3^{-1}/F(x_2 x_2^{-1})) \]

we are done.

Putting \( x_1 = 1, \ x_2 = s \) and \( x_3 = st \) gives \( \sup(sF(t)) = F(s) F(st/F(s)) .\)

If the Banach lattice is an abstract \( M \) space we have \( F(t) = \max(1,t) \).

Otherwise, as we now show, \( F \) is strictly increasing. We recall \( f \in \partial F(u) \)

means \( F(x) \geq F(u) + (f,x-u) \) for all \( x \).

Proposition 2.6. Either \( F(t) = \max(1,t) \) for all \( t \) or \( F \) is strictly increasing on \( [0,\infty) \), and if \( u > 0 \), \( f \in \partial F(u) \), then \( f > 0 \).

Proof. If \( F \) is not strictly increasing there is \( \epsilon > 0 \) with

\( F(\epsilon) = F(0) = 1 \), and \( F(t) > 1 \) for \( t > \epsilon \). For \( t > 1 \), \( F(\epsilon F(t)) = F(\epsilon t) > 1 \),
giving \( \epsilon F(t) = \epsilon t \) and \( F(t) = t \) for \( t > 1 \).

Hence \( F(t) = t \) for \( t \geq 1 \) and so \( F(t) = 1 \) for \( t < 1 \), giving \( F(t) = \sup(1,t) \).

If \( u > 0 \) and \( 0 \in \partial F(u) \) then \( F(u) \leq F(0) \) and \( F \) is not strictly increasing.
3. Proving $X$ is $\ell^3_p$ by solving the functional equation

In this section, we prove $F(t) = (1 + t^p)^{1/p}$, along the lines of Theorem 4 of [4, §15], but with complications.

As one would imagine, this can be translated into the form of Theorem 4 of [4, §15], and this is done in Theorem 3.3. Then we readily derive the conclusion that $X$ has $\ell^3_p$ norm.

**Theorem 3.1.** Let $F$ be a nondecreasing function on $[0,\infty)$ with $F(0) = 1$, $F(1) > 1$, and $F(s) > 0$ for all $s$. Suppose $F$ satisfies the equation $F(sF(t)) = F(s) F(st/F(s))$ for all $s$ and $t$. Then there is $p \in (0,\infty)$ such that $F(t) = (1 + t^p)^{1/p}$ for all $t \geq 0$. If $F(1) \leq 2$ then $p \geq 1$.

**Proof.** First we reduce the problem to the case $F(1) = 2$. Since $F(1) > 1$ there is $p \in (0,\infty)$ with $F(1) = 2^{1/p}$. If $F(1) \leq 2$ then $p \geq 1$. Let

$$G(t) = F(t^{1/p})^p, \quad (t \geq 0);$$

$$G(sG(t)) = (F(s^{1/p}F(t^{1/p})))^p$$

$$= F(s^{1/p})^p (F(s^{1/p} t^{1/p}/F(s^{1/p})))^p$$

$$= G(s) G(st/G(s)).$$

Note $G(1) = 2$, $G$ is nondecreasing, $G(0) = 1$, and $G(s) > 0$ for all $s$.

Let $G^{(n)}(t)$ denote the $n$th iterate of $G$ at $t$, so $G^{(n+1)}(t) = G(G^{(n)}(t))$. Let $a_n = G^{(n)}(0)$.

**Lemma 3.2.** For $m, n \geq 1$ we have $a_{nm} = a_m a_n$.

**Proof.** For $n = 2$, we have

$$a_{2m} = G(G(a_{2m-2}))$$

$$= G(1) G(a_{2m-2}/G(1))$$

$$= 2G(a_{2m-2}/2)$$

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so that if \( a_{2(m-1)} = a_2a_{m-1} = 2a_{m-1} \) then \( a_{2m} = a_2a_m = 2a_m \). Since \( a_{2m} = a_2a_m \) for \( m = 1 \) we have \( a_{2m} = a_2a_m \) for all \( m \).

Now fix \( j \geq 2 \) and suppose \( a_{nm} = a_na_m \) for all \( n < j \) and all \( m \). Then

\[
a_{jm} = G(m) (a_{(j-1)m})
= G(m) (a_m a_{j-1})
= G^{(m-1)}(G(a_m, G(a_{j-2})))
= G^{(m-1)}(G(a_m, G(a_{j-2}a_m/G(a_m))))
= G^{(m-1)}(a_{m+1}, G(a_{j-2}a_m/a_{m+1})).
\]

Now for \( 0 < k < m \),

\[
G^{(m-k)}(a_{m+k}, G(a_{j-2}a_m/a_{m+k})) = G^{(m-k-1)}(G(a_{m+k}, G(a_{j-2}a_m/a_{m+k})))
= G^{(m-k-1)}(G(a_{m+k}, G(a_{j-2}a_m/G(a_{m+k}))))
= G^{(m-k-1)}(a_{m+k+1}, G(a_{j-2}a_m/a_{m+k+1})).
\]

It follows that

\[
a_{jm} = a_{2m} \ G(a_{j-2}a_m/a_{2m})
= 2a_m \ G(a_{j-2}a_m/2a_m)
= 2a_m \ G(a_{j-2}/2)
= a_m \ G(G(a_{j-2}))
= a_ma_j.
\]

Thus by induction on \( j \) we have for all \( j \) that \( a_{jm} = a_ma_j \) for all \( m \), proving the lemma.
Proof of Theorem 3.1 continued. Now let \( m \) and \( n \) be positive integers. Then \( a_{(nm)} = (a_n)^m \) by using Lemma 3.2 \( m \) times. If \( j \) is a positive integer such that \( 2^{j-1} \leq nm < 2^j \) then \( 2^{j-1} = (a_2)^{j-1} \leq a_{(2^j-1)} \leq a_{(nm)} \leq a_{(2^j)} = (a_2)^j = 2^j \), since \( G \) is nondecreasing, so that

\[
j-1 \leq \log_2 (a_n)^m = m \log_2 a_n \leq j.
\]

Thus we have

\[
(j-1)/m \leq \log_2 a_n \leq j/m
\]

and also

\[
(j-1)/m \leq \log_2 n < j/m.
\]

Letting \( m \to \infty \) we have \( \log_2 (a_n) = \log_2 n \), so that \( a_n = n \) for all integers \( n > 0 \).

Thus \( G(n) = G(a_n) = a_{n+1} = n + 1 \) for integers \( n > 0 \).

Next suppose that \( m \) and \( j \) are positive integers and \( G(m/j) = 1 + m/j \). Then

\[
(j+1) G(m/(j+1)) = G(j) G(m/G(j))
\]

\[
= G(j) G(j(m/j)/G(j))
\]

\[
= G(j G(m/j))
\]

\[
= G(j(1+m/j))
\]

\[
= G(m+j).
\]

so that \( G(m/(j+1)) = (m+j+1)/(j+1) = 1 + m/(j+1) \). By induction on \( j \) we have \( G(r) = 1 + r \) for all rational \( r > 0 \). Finally by \( G \) being non-decreasing we have \( G(t) = 1 + t \) for all \( t > 0 \), giving \( F(t) = (1 + t^p)^{1/p} \).
We now restate this in the form of Bohnenblust's Theorem 4 of [4, §15] including the case $p = \infty$, and weakening the requirements.

Theorem 3.3. Let $f : [0,\infty] \times [0,\infty) \to [0,\infty)$ be given. There is $p \in (0,\infty]$ such that if $p < \infty$ then $f(a,b) = (a^p + b^p)^{1/p}$ and if $p = \infty$ then $f(a,b) = \sup(a,b)$, if and only if the following hold:

\begin{align*}
  f(1,0) &= 1 \quad (3.2) \\
  f(a,f(b,c)) &= f(f(a,b),c) \quad \text{for all } a, b, c \quad (3.3) \\
  f(ra,rb) &= rf(a,b) \quad \text{for } r, a, b \geq 0 \quad (3.4) \\
  f(1,t) &\text{ is not identically } 1 \quad (3.5) \\
  f(1,s) &\leq f(1,t) \quad \text{if } s \leq t \quad . \quad (3.6)
\end{align*}

Remark. We do not assume $f(a,b) = f(b,a)$. The condition 3.5 would follow from this assumption since we would have $f(1,t) = tf(t^{-1},1) = tf(1,t^{-1}) \geq tf(1,0) = t$. Nor do we assume $f$ to be continuous.

Proof of Theorem 3.3. Define $F : [0,\infty) \to [0,\infty)$ by $F(t) = f(1,t)$. By (3.2) and (3.6), $F(t) \geq F(0) = 1$ for $t \geq 0$. By (3.3) and (3.4),

\begin{align*}
  F(s)F(st/F(s)) &= F(s)f(1,st/F(s)) \\
  &= f(F(s),st) \\
  &= f(f(1,s),st) \\
  &= f(1,f(s,st)) \\
  &= f(1,sF(t)) \\
  &= F(sF(t)) , \quad (3.7)
\end{align*}

Suppose $F(1) = 1$. By (3.6), $F = 1$ on $[0,1]$. By (3.7) with $s = 1$, $F(F(t)) = F(t)$ for all $t$, and $F = I$ on the range $R(F)$ of $F$. We claim $F = I$ on $MR(F)$, the set of finite products of elements of $R(F)$,
by induction. For let $s \in R(F)^n$ and $F(t) \in R(F)$, and assume $F(s) = s$. Then $F(sF(t)) = sF(t)$ by (3.7) and $F = I$ on $R(F)^{n+1}$.

We claim 1 is a limit point of $R(F) \cap (1,\infty)$. If not, since $F$ is nondecreasing and $F = I$ on $R(F)$, either (a) $F(s) = 1$ on a neighbourhood of 1, or (b) by (3.6) $F(s) = \inf\{F(t) : F(t) > 1\} = K > 1$ on a right neighbourhood of 1, hence on $(1,K]$. Assuming (a), since $F(s) = F(t) = 1$ implies $F(st) = 1$ by (3.7), we have $F(s) = 1$ for all $s$, contradicting (3.5). Assuming (b), take $s \in (1,K)$ and $t_1 \in (1,K)$ with $st_1 < K$ and $t_2 \in (1,K)$ with $st_2 > K$. By (3.7),

$$K = K F(st_1/K) = F(sK) = K F(st_2/K) \geq K^2,$$

a contradiction.

Now if $A$ is a subset of $(1,\infty)$ closed under multiplication and having 1 as a limit point then $A$ is dense in $(1,\infty)$, for given $x > 1$ and $\varepsilon > 0$, take $a \in A$ with $xa < x+\varepsilon$, so that a power of $a$ is in the interval $(x, x+\varepsilon)$.

Hence $MR(F)$ is dense in $(1,\infty)$. Since $F$ is the identity on $MR(F)$, (3.6) gives $F = I$ on $(1,\infty)$, and $F(t) = \sup(1,t)$.

If $F(1) \neq 1$, then $F(1) > 1$ by (3.6), and by Theorem 3.1 there is $p \in (0,\infty)$ with $F(t) = (1 + t^p)^{1/p}$, giving $f(a,b) = (a^p + b^p)^{1/p}$ by (3.4).

We can now show that Ando's three dimensional result holds with only sublattices through $e_1$ or $e_3$ satisfying the basic hypothesis.

**Theorem 3.4.** Let $X$ be a three dimensional Banach lattice with unit vectors $e_1$, $e_2$ and $e_3$ such that $e_i \wedge e_j = 0$ for $i \neq j$. If every two dimensional vector sublattice which contains either $e_1$ or $e_3$ is the range of a contractive projection on $X$ then there is $p \in [1,\infty]$ such that for scalars $x_1$, $x_2$, $x_3$, if $p < \infty$ then $\|x_1e_1 + x_2e_2 + x_3e_3\| = (|x_1|^p + |x_2|^p + |x_3|^p)^{1/p}$ and if $p = \infty$ then $\|x_1e_1 + x_2e_2 + x_3e_3\| = \max(|x_1|, |x_2|, |x_3|)$. 

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Proof. If \( F(t) = \max(1,t) \) then for \( x_1, x_2, x_3 > 0 \) we have
\[
\|x_1e_1 + x_2e_2 + x_3e_3\| = x_1 F((x_2/x_1) F(x_3/x_2)) = \max(x_1, x_2, x_3),
\]
so we say \( p = \infty \). Otherwise by Proposition 2.6 \( F \) is strictly increasing. By Theorem 3.1 there is \( p \in [1,\infty) \) such that \( F(t) = (1 + t^p)^{1/p} \). For \( x_1, x_2, x_3 > 0 \)
\[
\|x_1e_1 + x_2e_2 + x_3e_3\| = x_1 F((x_2/x_1) F(x_3/x_2)) = (x_1^p + x_2^p + x_3^p)^{1/p}.
\]
By continuity of the norm and symmetry of the norm in a Banach lattice we have for scalars \( x_1, x_2, x_3 \)
\[
\|x_1e_1 + x_2e_2 + x_3e_3\| = (|x_1|^p + |x_2|^p + |x_3|^p)^{1/p} \quad (p < \infty)
\]
or \( \max(|x_1|, |x_2|, |x_3|) \quad (p = \infty) \).

4. Characterising \( \ell_p \) spaces by classes of sublattices being ranges of nonexpansive projections

In this section we characterise \( \ell_p(I) \) and \( \sigma_0(I) \), \( I \) an index set, applying our results on the three dimensional case. We require only that 2 dimensional sublattices through \( e_i \) should be ranges of contractive projections. Similarly, we obtain a \( p \) additive norm under this condition if we require that \( A \) is a set of not necessarily disjoint positive elements such that for \( e, f \in A \), \( (e-f)^+ \in A \) and \( (e-f)^+ \wedge f = 0 \).

A further application of the three dimensional results is the characterisation of \( \ell_p \) direct sums. Note that Lemma 4.1 and Theorem 4.2 may be weakened slightly by omitting "\( i \neq 1 " \), but Theorem 3.4 rather than Ando's result is required in the proof still.

Lemma 4.1. Suppose \( e \ldots e_n, n \geq 3 \), are vectors in a Banach lattice \( X \), with \( e_i \wedge e_j = 0 \) if \( i \neq j \). Suppose any two dimensional sublattice through \( e_i \), \( i \neq 1 \), is the range of a contractive projection. Then there is \( p \in [1,\infty] \) such that the norm is \( p \) additive on \( \text{span}\{e_i\} \).

Proof. By Theorem 3.4 there is \( p \in [1,\infty] \) with the norm \( p \) additive on \( \text{span}\{e_1, e_2, e_3\} \). Suppose \( n \geq 4 \) and assume the norm is \( p \) additive on \( \text{span}\{e_1 \ldots e_{n-1}\} \). Assume \( \|e_i\| = 1 \) for all \( i \).

Now there is a \( p' \) summable norm on \( \text{span}\{e_{n-2}, e_{n-1}, e_n\} \), for some \( p' \in [1,\infty] \). Since \( \|e_{n-2} + e_{n-1}\| = 2^{1/p'} \) and \( \|e_{n-2} + e_{n-1}\| = 2^{1/p} \), \( p' = p \).
Given scalars $x_1 \ldots x_n > 0$, $\text{span}\{x_1 e_1 + \cdots + x_{n-2} e_{n-2}, e_{n-1}, e_n\}$ has a $p^*$ summable norm, for some $p^* \in [1, \infty]$, by Theorem 3.4. Since $2^{1/p^*} = \|e_{n-1} + e_n\| = 2^{1/p}$, $p^* = p$. Hence

$$\|x_1 e_1 \ldots + x_{n-1} e_{n-1} + x_n e_n\|$$

$$= \left(\|x_1 e_1 \ldots + x_{n-1} e_{n-1}\|^p + |x_n|^p\right)^{1/p} \quad \text{(since } p^* = p)$$

$$= \left(|x_1|^p \ldots + |x_{n-1}| + |x_n|^p\right)^{1/p}$$

by the inductive hypothesis, where this has the usual interpretation if $p = \infty$.

Theorem 4.2. Let $X$ be a Banach lattice and $A = \{e_i\}$ a set of elements of $X$ with $e_i \wedge e_j = 0$ for $i \neq j$. Suppose the span of $A$ is dense, that $1 \in I$ and if $i \neq 1$ then any two dimensional sublattice through $e_i$ is the range of a contractive projection. Then $X$ is linearly isometric and lattice isomorphic to $c_0(I)$ or $\ell_p(I)$ for some $p \in [1, \infty]$.

Proof. By Lemma 4.1 there is $p \in [1, \infty]$ such that the norm on the linear span of $A$ is $p$-additive. Assuming without loss of generality that $\|e_i\| = 1$ for all $i$, we define $T : \text{span}(A) \rightarrow \ell_p(I)$ by letting $Te_i$ be 1 at $i$ and 0 elsewhere and extending linearly. Hence $\|Tx\|_p = \|x\|$ for all $x$ in $\text{span}(A)$. As the range of $T$ is dense in $\ell_p(I)$, or $c_0(I)$ if $p = \infty$, $T$ extends uniquely to a linear isometry and lattice isomorphism of $X$ onto $\ell_p(I)$ or $c_0(I)$.

Example 4.3. We note that it is not sufficient to assume merely that there are contractive projections onto two dimensional sublattices through $e_i$, $i \neq 1$ or 2.

For let $X = \ell^p$ with $\left\| \sum_{i=1}^u x_i e_i \right\|^p = (|x_1|^q + |x_2|^q)^{p/q} + |x_3|^p + |x_4|^p$ where $p \neq q$. Any two dimensional sublattice through $e_3$ or $e_4$ is the range of a contractive projection, but the norm is not $p$ additive.

Remark. In the following result, Theorem 4.4., we have in mind $A$ being a set of characteristic functions of measurable sets.
Theorem 4.4. Let $X$ be a Banach lattice and suppose $A = \{e_i\}_{i \in I}$ is a set of positive elements of $X$ such that $e_i, f \in A$ implies $(e-f)^+ \in A$, and $(e-f)^+ \wedge f = 0$. Suppose $X$ is the closed span of $A$, and the span of $A$ has dimension $\geq 3$. Suppose that any two-dimensional sublattice containing any $e_i, i \in I, i \neq 1$, is the range of a contractive projection. Then the norm on $X$ is $p$ additive for some $p$.

Proof. We claim that given $e_1 \ldots e_m \in A$ there exists a finite set
\[\{g_1 \ldots g_n\} \subseteq A, \quad g_i \wedge g_j = 0 \text{ if } i \neq j, \text{ with } e_i \in \text{span}\{g_1 \ldots g_n\}.\]

We assume that this holds for $e_1 \ldots e_{m-1}$. That is, there is
\[\{g_1 \ldots g_k\} \subseteq A, \quad g_i \wedge g_j = 0 \text{ if } i \neq j, \text{ with } e_1 \ldots e_{m-1} \in \text{span}\{g_1 \ldots g_k\}.\]

For $i = 1 \ldots k$, let $g_i^1 = g_i \wedge e_m = (g_i - (g_i - e_m)^+) \in A$. Also let $g_i^2 = (g_i - e_m)^+ \in A$. By assumption $g_i^1 \wedge g_i^2 = 0$. Now $0 \leq g_i^2$, $g_i^1 \leq g_i$, so $g_i^p \wedge g_j^p = 0$ for $p, q = 1$ or $2$ and $i \neq j$.

Define $g_i^3 = (e_m - g_i)^+$ and define $g_i^3$ inductively for $2 \leq i \leq k$ $g_i^3 = (g_{i-1}^3 - g_i)^+$. We claim $e_m = \bigoplus_{j=1}^{k} g_j^1 + g_k^3$. For $e_m = g_1^1 + g_1^3$, and assuming $1 \leq i < k$ and $e_m = \bigoplus_{j=1}^{i} g_j^1 + g_k^3$, we have
\[
e_m = \bigoplus_{j=1}^{i} g_j^1 + (g_i^3 - g_{i+1})^+ \wedge g_{i+1} \]
\[= \bigoplus_{j=1}^{i} g_j^1 + (g_i^3 - g_{i+1})^+ + e_m \wedge g_{i+1},\]
proving the claim by induction.

Hence $\{e_1 \ldots e_m\} \subseteq \text{span} \cup \{g_i^1, g_i^2, g_k^3 : 1 \leq i \leq k\}$ and these are disjoint. Since $\text{dim} (\text{span}(A)) \geq 3$, by Lemma 4.1. there is $p$ such that
the norm is $p$ additive on $\text{span } U \{g_i^* : 1 \leq i \leq n\}$ where $g_i^* \in A$, $g_i^*$ disjoint. This $p$ is independent of such a set $\{g_i^* : 1 \leq i \leq n\}$ and the norm is $p$ additive on the span of $A$.

By Exercise 4, §15 of [4], the completion of a normed lattice with $p$ additive norm has $p$ additive norm. Hence $X$ has $p$ additive norm.

Now we look at a space which is not an $L_p$ space, merely an $\ell_p$ product of an ideal with $p$ additive norm with another ideal. First we need a result proved in the last 3 paragraphs of Theorem 4 of §16 of Lacey's book [4].

**Lemma 4.5.** Let $E$ be a Banach lattice of dimension $\geq 3$ such that for any three mutually disjoint positive elements $x_1$, $x_2$, and $x_3$ there is $p \in [1, \infty]$ such that the norm on the span of $\{x_1, x_2, x_3\}$ is $p$ additive. Then there is $p \in [1, \infty]$ such that the norm on $E$ is $p$ additive.

**Theorem 4.6.** Let a Banach lattice $E$ be the direct sum of ideals $A$ and $B$ with $\dim(A) \geq 2$ and $\dim(E) \geq 3$. Then there exists $p \in [1, \infty]$ such that the norm on $A$ is $p$-additive and $E$ is the $\ell_p$ direct sum of $A$ and $B$ if and only if for any two dimensional sublattice $X$ of $E$ with $X \cap A \neq \{0\}$, $X$ is the range of a (positive) contractive projection on $E$.

**Proof.** Let $A$ have $p$ additive norm and let $E$ be the $\ell_p$ direct sum. Given a two dimensional sublattice $X$ of $E$ with $0 \neq a \in A \cap X$, suppose $X$ is spanned by $a$ and $y$. If $y = u + z$ with $u \in A$ and $z \in B$ then $X$ is contained in the Banach lattice $Y$ spanned by $A$ and $z$ and we may suppose $z \geq 0$. The norm on $Y$ has to be $p$ additive, and so there is a positive contractive projection $P$ of $Y$ onto the two dimensional sublattice $X$. Since there is a positive contraction of $B$ onto $\text{span}\{z\}$ and there is a positive contractive projection $Q$ of $E$ onto $Y$, $PQ$ is the required projection.

Conversely suppose every two dimensional sublattice $X$ of $E$ with $X \cap A \neq \{0\}$ is the range of a contractive projection. We may assume
dim(B) ≥ 1. Let \( b \in B, \ b \neq 0 \). For each 2-dimensional sublattice \( Z \) of \( A \), \( Y = (Z, \|b\|) \) is a 3-dimensional sublattice of \( E \). Let \( e_1 \) and \( e_3 \) be mutually disjoint positive unit vectors in \( Z \). Any two dimensional sublattice of \( Y \) which contains \( e_1 \) or \( e_3 \) is the range of a positive contractive projection on \( Y \). By Theorem 3.4 there is \( p \in [1, \infty) \) such that the norm on \( Y \) is \( p \)-additive. Similarly if \( e_1 \), \( e_2 \) and \( e_3 \) are mutually disjoint positive unit vectors in \( A \) then the norm on \( \text{span}\{e_1, e_2, e_3\} \) is \( p \)-additive. By Lemma 4.5 the norm on \( \text{span}(A, ||b||) \) is \( p \)-additive for some \( p \in [1, \infty) \). Thus \( A \) has \( p \)-additive norm and if \( a \in A \) then \( \|a + b\| = \|a - b\| = \|a\| + |b|\) so \( \|a + b\| = (\|a\|^p + \|b\|^p)^{1/p} \), or \( \|a + b\| = \max(\|a\|, \|b\|) \) if \( p = \infty \). Now since \( A \) has \( p \)-additive norm and at least 2 dimensions, \( p \) is independent of \( b \) so that \( E \) is the \( \ell_p \) direct sum of \( A \) and \( B \).

Corollary 4.7. Let a Banach lattice \( E \) be the direct sum of ideals \( A \) and \( B \), with \( \text{dim}(E) \geq 3 \). There is \( p \in [1, \infty) \) such that the norm on \( E \) is \( p \)-additive if and only if for any two dimensional sublattice \( X \) of \( E \) with \( X \cap A \neq \{0\} \) or \( X \cap B \neq \{0\} \), \( X \) is the range of a contractive projection on \( E \).

Theorem 4.8. Let \( E \) be a Banach lattice and \( \{E_i : i \in I\} \) a set of ideals such that \( E_i \cap E_j = \{0\}, \ i \neq j \), and the span of \( \cup\{E_i : i \in I\} \) is dense in \( E \). Let \( 1 \in I \), \( \text{dim}(E) \geq 3 \), \( \sum_{i \neq 1} \text{dim}(E_i) \geq 2 \). Suppose any two \( 1 \)-dimensional sublattice \( X \) with \( X \cap E_i \neq \{0\} \) for some \( i \neq 1 \) is the range of a contractive projection.

Then there is a \( p \in [1, \infty) \) such that the closed span \( Y \) of \( \cup\{E_i : i \neq 1\} \) has \( p \)-additive norm and \( E \) is the \( \ell_p \) direct sum of \( E_1 \) and \( Y \). Thus \( E \) is the \( \ell_p(1 \leq p < \infty) \) or \( c_0 \) direct sum of the ideals \( E_i \), \( i \in I \).

Proof. We claim the norm is \( p \)-additive on \( A \), the span of \( \cup\{E_i : i \neq 1\} \). We may assume at least two \( E_i \) are nonzero.

Suppose only two \( E_i \) are nonzero. The claim follows by Corollary 4.7 if \( E_1 \) is zero and by Theorem 4.6 if \( E_1 \) is nonzero.
Suppose at least three $E_i$'s are nonzero. By Lemma 4.1 there is $p$ such that the norm is $p$ additive on the span of $\{e_1, \ldots, e_n\}$ for $e_i \in \{E_i : i \in I\}$, $e_i \wedge e_j = 0$ if $i \neq j$, and at most one $e_i$ in $E_1$. This proves the claim.

By §15 Example 4 of [4] the norm is $p$ additive on $Y$, the closure of $A$. By Theorem 4.6, $E$ is the $\ell_p$ direct sum of $E_1$ and $Y$.

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