1. Introduction

The study of Geometric Function Theory is one of the most fascinating aspects of the theory of analytic functions of a complex variable. In this field, we are mainly concerned with the power series of the form

\[ f(z) = b_0 + b_1 z + b_2 z^2 + \ldots + b_n z^n + \ldots \]

in the complex variable \( z \) that are convergent in a domain \( D \). Such a power series may be interpreted as a mapping of the domain \( D \) in the \( z \)-plane onto some range set \( E \) in the \( w \)-plane. There are two natural questions to ask: (i) given the sequence \( \{b_0, b_1, \ldots \} \) of coefficients, what can be said about the geometry of the range set \( E \); and (ii) given some geometric property of the range set \( E \), what can be said about the sequence \( \{b_0, b_1, \ldots \} \)? A nice geometric property from the point of view of conformal mapping possessed by an analytic function \( f(z) \) is that of univalence in \( D \). We recall that a function \( f(z) \) that is analytic in \( D \) is said to be univalent in \( D \), if it never takes the same value twice, that is, \( f(z_1) \neq f(z_2) \) if \( z_1 \neq z_2 \), whenever \( z_1, z_2 \in D \).

Univalent functions are the simplest analytic functions from the geometric point of view. The theory of univalent functions is so vast and complicated that certain simplifying assumptions are necessary. First is to take the unit disk \( \Delta = \{z : |z| < 1\} \) in place of arbitrary domain \( D \). Second is to take normalization conditions: \( f(0) = 0, f'(0) = 1 \).
With these assumptions, we can rewrite $f(z)$ in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \Delta. \quad (1)$$

It may be noted that the normalization does not disturb the univalence of the function because if $f(z)$ is analytic and univalent in $\Delta$, so also is the function

$$g(z) = \frac{f(z) - f(0)}{f'(0)}.$$

We may also add that $f'(0) \neq 0$. If $f'(0)$ vanishes in $\Delta$, then $f(z)$ could not be univalent in $\Delta$. Further, let

$$U = \{f : f \text{ is analytic and normalized in } \Delta\}$$

and

$$S = \{f : f \in U \text{ and } f \text{ is univalent in } \Delta\}.$$

Example 1. It is easy to see that the function $g(z) = (1+z)/(1-z)$ is univalent in $\Delta$, and $g(\Delta)$ is the half-plane $\text{Re } g(z) > 0$.

Example 2. One can prove that the function

$$k(z) = \frac{1}{4} \left[ \left( \frac{1+z}{1-z} \right)^2 - 1 \right] = \frac{z}{(1-z)^2} \quad (2)$$

is univalent in $\Delta$. This is called the Koeb function and it maps $\Delta$ onto the entire complex plane except the slit along the negative real axis from $-\infty$ to $-1/4$. In an intuitive sense, the Koeb function is the largest function in the family $S$ because we cannot adjoin to $k(\Delta)$ any open set without destroying univalence. (This is not the only domain with this property. However, the maximal nature of the domain $k(\Delta)$ is its symmetry and the size of the coefficients of $k(z)$.) The functions
\[ e^{-i\Theta} \kappa(e^{i\Theta} z) = \frac{z}{(1-e^{i\Theta} z)^2} = z + \sum_{n=2}^{\infty} ne^{i(n-1)\Theta} z^n \quad (3) \]

also belong to \( S \) and are referred to as the rotations of the Koebe function.

The serious study of univalent functions began in 1907, when Koebe published a paper [58] in which he proved the existence of a positive constant \( c \) such that

\[ \bigcap_{f \in \mathcal{S}} f(\Delta) \supset \{ w : |w| \leq c \} . \]

In 1916, Bieberbach [13] proved that \( c = 1/4 \). Thus we have an interesting result that the open disk \( |w| < 1/4 \) is always covered by the map of \( \Delta \) of any function \( f \in S \). Furthermore, Bieberbach observed that the Koebe function or any of its rotations are the only ones whose image domain contains a boundary point on the circle \( |w| = 1/4 \). In the same paper, he proved that the absolute value of the second coefficient of any function \( f \) in \( S \) is never more than \( 2 \). He further observed that for the Koebe function, we have \( |a_n| = n \geq 2 \). By proving a distortion theorem, Koebe showed that the class \( S \) is compact. Thus for every positive integer \( n \), there is a uniform bound on the magnitude of the coefficient of \( z^n \) for all \( f \) in \( S \). These observations encouraged Bieberbach to make the famous conjecture [13] in 1916.

The Bieberbach Conjecture. Among all the functions

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \Delta) \]

in \( S \), the Koebe function has the largest coefficients, that is

\[ |a_n| \leq n, \quad n \geq 2 \]

for all \( f \in S \).

Ludwig Bieberbach, a German mathematician, is well known in the mathematics community. His conjecture has challenged the best investigators
of the world for about 70 years. This conjecture has been considered so
difficult to prove or disprove that some eminent mathematicians believed it
to be false. A great number of researchers have devoted their careers in
trying to resolve it. The Bieberbach conjecture has inspired several
developments in geometric function theory by imparting many powerful new
methods and generating a large number of related problems, some of which are
open while others have been solved completely. In all likelihood, it has
already contributed more to mathematics as a challenging problem than it will
ever contribute as a theorem.

In the mid-summer of 1984, Louis de Branges from the University of
Purdue greatly excited and surprised the mathematics world [59] by making a
claim that he had resolved the Bieberbach conjecture. Thus he ended the
efforts of many mathematicians of almost seventy years.

In this expository survey, we mention only those results of the theory
of univalent functions which bear directly on the Bieberbach conjecture or
on the problems resulting from an effort to resolve the conjecture. We shall
also emphasize some recent results and related open problems. Since there
is an immense literature [12,20,39], books [41,54,74,85,97] and several
surveys [24,40,57] on theory of univalent functions, we shall make a
selection of the results relevant to our precise objective.

2. The Bieberbach conjecture for individual coefficients

Let \( \mathbb{U} \) be the class of functions

\[
g(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots
\]

analytic and univalent in \(|z| > 1\) except for a simple pole at \(\infty\) with
residue 1. Grownwall [43] in 1914 proved that \( \sum_{n=1}^{\infty} n |b_n|^2 \leq 1 \), whenever
g \( \in \mathbb{U} \). This elementary result is known as the area theorem. Applying
this result to the function

\[
g(z) = (f(1/z^2))^{-1} = z - a_2/2z + \ldots,
\]

one can easily prove the Bieberbach conjecture for \( n = 2 \). There are many
simple proofs available to prove the conjecture for \( n = 2 \).

In 1923, Löwner [68] proved the conjecture for \( n = 3 \) by using the type of functions which map the disk \( \Delta \) onto the full plane slit along some Jordan arc terminating at \( \infty \). These functions are dense in \( S \) in the topology of uniform convergence on compact subsets of \( \Delta \). Löwner showed that they can be generated (along with certain other univalent functions) by a differential equation of prescribed form. This leads to a parametric representation of the coefficient \( a_3 \) in terms of integrals of a function \( k(t) \), \( |k(t)| = 1 \) given by

\[
a_3 = -2 \int_0^\infty e^{-2t} |k(t)|^2 \, dt - 4 \left( \int_0^\infty e^{-t} k(t) \, dt \right)^2.
\]

It can be shown from (4) that \( |a_3| \leq 3 \) with equality only for the Koebe function. By using variational methods, Schaeffer and Spencer [96] in 1943 gave another proof of \( |a_3| \leq 3 \).

Schiffer [98,99,100] developed a Calculus of variations for the class \( S \) and had used it to prove that each \( f \in S \) which maximizes \( |a_n| \) must map \( \Delta \) onto the plane slit along a system of analytic arcs satisfying a certain differential equation. In 1955, Garabedian and Schiffer [33] used the variational method developed by Schiffer to establish the Bieberbach conjecture for \( n = 4 \). Their proof was extremely complicated. However, in 1960, Charzynski and Schiffer [21] provided an elementary and simple proof of \( |a_4| \leq 4 \). Their proof was based on the Grunsky inequalities [44] which can be established by a slight generalization of the method used to prove the area theorem.

The Bieberbach conjecture for \( n = 6 \) was proved in 1968 by Pederson [81] and Ozawa [77,78]. These researchers have used Grunsky inequalities [44] to prove that \( |a_6| \leq 6 \). Their proof is difficult and too long.

Applying a variational method, Garabedian and Schiffer 1967 [34] generalized the Grunsky inequalities to yield certain relations known as the Garabedian - Schiffer inequalities. The inequalities involve auxiliary parameters which can be choosen to optimize the estimates. Using the Garabedian - Schiffer inequalities, Pederson and Schiffer [83] in 1972
settled the conjecture for the fifth coefficient.

No other case of the Bieberbach conjecture has been verified. However, Ozawa and Kubota 1972 [79] have shown that \( \text{Re } a_8 \leq 8 \) provided that \( \text{Re } a_2 \geq 0 \). By using a computer, Horowitz 1977 [50] has shown that if \( p(z) = z + \sum_{k=1}^{m} a_k z^k \) is a polynomial in \( S \) of degree not greater than 27, then \( |a_k| \leq k, \ 2 \leq k \leq 27 \).

3. Some estimates for all the coefficients

In the meantime, mathematicians got closer and closer to the Bieberbach estimate for all the coefficients. The first good estimate for all the coefficients was given by Littlewood [65] in 1925. By using the crude estimation of the Cauchy integral formula for \( \alpha \) of

\[
|a_n| \leq r^{-n} M_1(r, \alpha), \quad 0 < r < 1,
\]

where

\[
M_1(r, \alpha) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| d\theta,
\]

he proved that \( |a_n| < en \) for all \( n \), where \( e \) is the base of the natural logarithms. In 1951, Bazilevič [11] improved Littlewood's estimate for \( M_1(r, \alpha) \) to prove that \( |a_n| < en/2 + 1.51, \ n = 2, 3, \ldots \). Baernstein 1974 [9] proved that \( |a_n| < en/2, \ n = 2, 3, \ldots \).

Milin [72] developed a new method in 1965 to obtain information from the Grunsky inequality and thus proved that \( |a_n| < 1.243 n, \ n \geq 2 \). His result was improved by FitzGerald [30] in 1972. Using a method involving, roughly speaking, exponentiation of the Grunsky inequalities [44], he obtained the inequalities

\[
\sum_{k=1}^{n} k|a_k|^2 + \sum_{k=n+1}^{2n-1} (2n-k)|a_k|^2 \geq |a_n|^4, \quad n = 2, 3, \ldots. \quad (5)
\]
FitzGerald [30] used the inequalities (5) to prove

$$|a_n| < (7/6)^n < 1.081n, \quad n \geq 2.$$  

Horowitz 1978 [49] further improved the estimate of FitzGerald by using a stronger form of (5), again due to FitzGerald [30]. His estimate was

$$|a_n| \leq (1,659,164,137/681,080,400)^{1/14} n \approx 1.0657n.$$  

The Bieberbach conjecture is true for functions with sufficiently small second coefficient. We list the estimates on the second coefficient as follows:

For each function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $S$, $|a_n| < n$ for all $n \geq 2$ if

- $|a_2| < 1.05$, Aharonov 1970 [1] and Illina 1973 [52]
- $|a_2| < 1.15$, Ehrig 1974 [28]
- $|a_2| < 1.55$, Bishouty 1976 [15]
- $|a_2| < 1.61$, Bishouty 1977 (unpublished).

4. Bieberbach Conjecture for subclasses of $S$

Failure to settle the Bieberbach conjecture for all the coefficients has led many researchers to introduce and investigate properties of several subclasses of $S$. We give below some of the most important subclasses of $S$.

Convex univalent functions. These are the functions which map the unit disk $\Delta$ onto the convex domains. A domain $E$ is said to be convex if the line segment joining any two points of $E$ lies wholly in $E$. Denote the class of all convex univalent functions in $\Delta$ by $K$. These functions were first studied by E. Study [106] in 1913. Robertson [87], in 1936, observed that function $f \in S$ is convex in $\Delta$ if and only if $\text{Re}(1+f''(z)/f'(z)) > 0$, $z \in \Delta$. Robertson [87] further proved that $|a_n| \leq n$, $n \geq 2$ for functions in the class $K$. In fact, we have $|a_n| \leq 1$ for all $f \in K$ and
\[ z/(1-z) = z + \sum_{n=2}^{\infty} z^n \] is an extremal function. Thus the Bieberbach estimate of \(|a_n|\) for \(K\) can be lowered.

Starlike univalent functions. Another important subclass of \(S\) consists of the starlike functions first treated by Alexander [7] in 1916. These functions map \(\Delta\) onto a domain starshaped with respect to the origin. A domain \(E\) containing the origin \(w = 0\) is said to be starshaped (or starlike) with respect to the origin if the line segment joining \(w = 0\) to any other point of \(E\) lies completely in \(E\). Denote the class of all starlike univalent functions in \(\Delta\) by \(S^*\). The Koebe function \(z/(1-z)^2\) is starlike because it maps \(\Delta\) onto the entire complex plane minus the slit \(-\infty < w \leq -1/4\). Robertson [87] showed that \(f \in S^*\) if and only if \(\Re\{\eta f'(z)/f(z)\} > 0\), \(z \in \Delta\). The inequalities \(|a_n| \leq n\), \(n \geq 2\) hold for the class \(S^*\) and are sharp for the Koebe function [87].

Spiral-like univalent functions. A class wider than the class \(S^*\) is the class of spiral-like functions introduced by Špaček [105] in 1932. An analytic normalized function \(f\) is said to be spiral-like function in \(\Delta\) if \(\Re\{\eta f'(z)/f(z)\} > 0\), \(z \in \Delta\) for some \(\eta\) such that \(|\eta| = 1\). Taking \(\eta = \exp(i\lambda)\), \(|\lambda| \leq \pi/2\), Libera [64] in 1967 introduced the class \(H^\lambda\) of all \(\lambda\)-spiral-like functions in \(\Delta\). Špaček [105] proved that \(H^\lambda \subset S\) and that the Bieberbach conjecture holds for \(H^\lambda\).

Univalent functions with real coefficients. Let \(SR\) denote the subset of functions in \(S\) for which all the coefficients \(a_n\) of \(f\) are real. The Bieberbach conjecture holds for \(SR\) and it was proved by Dieudonné [22] in 1931.

Close-to-convex functions. In 1952, Kaplan [56] introduced the class \(C\) of all close-to-convex functions in \(\Delta\). A function \(f\), analytic in \(\Delta\), is said to be close-to-convex in \(\Delta\) if there exists a starlike function \(g \in S^*\) such that

\[
\Re\left\{\frac{zf'(z)}{g(z)}\right\} > 0, \quad z \in \Delta.
\]
Geometrically, the condition (6) implies that the image of each circle $|z| = r < 1$ is a curve with the property that as $\theta$ increases, the angle of the tangent vector does not decrease by more than $-\pi$ in any interval $[\theta_1, \theta_2]$. Thus the curve cannot make a "hairpin bend" backward to intersect itself.

Kaplan [56] proved that $C \subset S$. Reade [86] in 1955, proved that the Bieberbach conjecture holds for the family $C$.

Typically real functions. If the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic in $\Delta$ and if for every non-real $z$ in $\Delta$

$$(\text{Im} z)(\text{Im} f(z)) \geq 0,$$

then $f$ is said to be a typically real function in $\Delta$. This merely means that the upper half of $\Delta$ must go into the upper half-plane under $f$, and the lower half of $\Delta$ must go into the lower half-plane under $f$; the interval $(-1,1)$ goes into the real axis. Denote the class of all the typically real functions in $\Delta$ by $TR$. The class $TR$ was introduced by Rogosinski [93] in 1932. He also proved the Bieberbach conjecture for $TR$.

From the definitions of the above subclasses of univalent functions, it follows that

$$K \subset S^* \subset C \subset S, \ K \subset S^* \subset H^\lambda \subset S$$

and

$$SR \subset TR \subset S.$$
5. Local form of Bieberbach Conjecture

The local Bieberbach conjecture states that the Koebe function yields the maximum of $\Re\{a_n\}$ at least for those functions $f$ in the family $S$ which are close enough to it in some appropriate topology. There are several interesting results on the local form of the Bieberbach conjecture. Using the Grunsky inequality, Garabedian, Ross and Schiffer 1965 [35] proved the local Bieberbach conjecture for the case of even index $n = 2m$. In 1967, Garabedian and Schiffer [34] derived a generalization of the Grunsky inequality through a differential equation of the form

$$[f''(z)]^2 E\left(\frac{1}{f'}\right) = R(z),$$

where $E(t)$ is almost quadratic and $R(z)$ is a rational function; and they used generalized Grunsky inequality to prove

Theorem. [34] If a function $f$ in $S$ is close enough to the Koebe function, for example, if $|a_2-2| < \epsilon_m$ for an appropriate small number $\epsilon_m > 0$, then the Bieberbach conjecture is true for the odd coefficients $a_{2m+1}$ of $f(z)$, with equality holding for the Koebe function.

Using the Löwner method together with the variational method developed by Duren and Schiffer 1962 [27], Bombieri 1967 [16] proved the local Bieberbach conjecture in the following interesting form

$$\Re a_n < \begin{cases} n - a_n(2-\Re a_2) & \text{if } n \text{ even, } |2-a_2| < \delta_n \\ n - a_n(3-\Re a_3) & \text{if } n \text{ odd, } |3-a_3| < \delta_n \end{cases},$$

where $a_n$ and $\delta_n$ are some positive constants. In 1969, Pederson [82] established the equivalence of various topologies near the Koebe function.

6. Successive coefficients of univalent functions

Let $d_n = \|a_{n+1}\| - |a_n|$, $n = 2, 3, \ldots$ denote the difference of the moduli of successive coefficients of functions in $S$. The problem of estimating $d_n$ has attracted the attention of many researchers. The following results have been obtained:
\[ d_n = 0(n^{1/4} \log n), \quad \text{Goluzin 1946 [37]} \]
\[ d_n = 0((\log n)^{3/2}), \quad \text{Biernacki 1956 [14]} \]
\[ d_n < K \text{ for some absolute constant, Hayman 1963 [47]} \]
\[ d_n < 9, \quad \text{Milin 1968 [73]} \]
\[ d_n < 4.17, \quad \text{Ilina 1968 [51]}. \]

Milin [74] was able to obtain bounds for the moduli \( d_n \) for coefficients of odd univalent functions more precisely.

Leung 1978 [61] proved a conjecture made by Pommerenke [84, Problem 3.5]; namely,

\[ \|a_{n+1}\| - |a_n| \leq 1, \quad \forall \zeta \in S^* . \]

He further observed that equality occurs for fixed \( n \) only for the functions \( z/(1-\gamma z)(1-\zeta z) \) for some \( \gamma \) and \( \zeta \) with \( |\gamma| = |\zeta| = 1 \). Robertson [90] showed

\[ |m|a_m| - n|a_n| \leq |m^2-n^2| \]

for the class \( C \) with \( m-n \) being an even integer. It has been shown by Leung 1979 [62] that (7) holds for the class \( C \) for all positive integers \( n \) and \( m \). However, Jenkins 1968 [55] showed that the inequality (7) cannot hold for the whole class \( S \). For the class \( S \), Duren 1979 [25] has shown that

\[ \|a_{n+1}\| - |a_n| \leq e^\delta a^{-\frac{1}{2}} < 1.37a^{-\frac{1}{2}}, \quad n \geq 2 \]

with Hayman index \( \alpha > 0 \), where \( \delta < 0.312 \) is the Milin's constant.

7. Various conjectures related to Bieberbach conjecture

We now investigate several conjectures given so far to resolve the Bieberbach conjecture. Our inability to prove the Bieberbach conjecture for the class \( S \) suggested that perhaps we should attack somewhat weaker conjectures.
Littlewood conjecture (1925). If \( f \in S \) and \( f(z) \neq w \) for some complex number \( w \), then

\[
|a_n| \leq 4|w|n, \quad n \geq 2 .
\]  

(8)

It is well known that for such an omitted complex number \( w \), we have \( |w| \geq 1/4 \) and that if \( |w| = 1/4 \), then \( f \) must be the Koebe function or its rotation, and in this case (8) is true for all \( n \). On the other hand, if \( |w| > 1/4 \), then the right side of (8) exceeds \( n \) and hence the Littlewood conjecture is weaker than the Bieberbach conjecture. The above conjecture was proposed by Littlewood [65] in 1925.

Littlewood - Paley conjecture (1932). For each odd function

\[
g(z) = [f(z^2)]^{1/2} = z + c_3z^3 + ...
\]

in the class \( S \), \( |c_n| \leq 1 \) for \( n = 3, 5, \ldots \).

This conjecture was proposed by Littlewood and Paley [66] in 1932. In the same paper, they showed that \( |c_n| \leq A \) for all \( n \), where \( A \) is an absolute constant (their method gives \( A < 14 \)). If we apply their conjecture to the equation \( f(z^2) = [g(z)]^2 \), \( f \in S \), it easily implies the Bieberbach conjecture. The Littlewood-Paley conjecture is true for the class \( S^* \) [66]. However, the conjecture is false in general because Fekete and Szego 1933 [29] obtained the sharp inequality \( |c_5| \leq 1/2 + e^{-2/3} = 1.013 \). Using a variational method, Schaeffer and Spencer 1943 [96] proved that the equality \( |c_5| = 1.013 \) is attained for odd functions with real coefficients. Moreover, for odd functions in \( S \) with real coefficients, Leeman 1976 [60] has shown that \( |c_7| \leq 1090/1083 \), and this bound is sharp.

Robertson conjecture (1936). If \( g(z) = z + b_3z^3 + \ldots + b_{2k-1}z^{2k-1} + \ldots \) is any odd function in \( S \), then

\[
1 + \sum_{k=2}^{n} |b_{2k-1}|^2 \leq n, \quad n \geq 2 .
\]
If we apply the Robertson conjecture to the odd function \[ (g(z))^2 = f(z^2) \], the Bieberbach conjecture immediately follows. The above conjecture was proposed by Robertson [88] in 1936. For \( n = 2 \), the conjecture is equivalent to \( |a_2| \leq 2 \). Using the Löwner's variational method, Robertson [88] proved his conjecture for \( n = 3 \). After a period of thirty-four years, Friedland 1970 [32] proved the conjecture for \( n = 4 \). He used the Grunsky inequalities.

Rogosinski conjecture (1943). If a function \( g(z) = b_1z + b_2z^2 + \ldots \), analytic in \( A \), is subordinate to a function \( f \) belonging to \( S \), then \( |b_n| \leq n, n \geq 1 \).

This conjecture was advanced by Rogosinski [95] in 1943. Since any function is subordinate to itself, it follows that the Rogosinski conjecture implies the Bieberbach conjecture. For \( n = 1 \), the Rogosinski conjecture is contained in the Schwarz lemma. Earlier, Littlewood [65] proved it for \( n = 2 \). Rogosinski [95] proved it for all \( n \) for the classes \( S^* \), \( K \), and \( SR \). Robertson 1965 [89] established this conjecture for \( C \). In 1970, Robertson [92] observed that the Rogosinski conjecture is implied by the Robertson conjecture.

Mandelbrojt-Schiffer conjecture. If \( f \) and \( g \) are in \( S \), then the function \( J \), defined by
\[
J(z) = f(z) * g(z) = z + \sum_{n=2}^{\infty} \frac{a_n b_n}{n} z^n,
\]
is in \( S \).

If the above conjecture was true, we could immediately prove the Bieberbach conjecture. In fact, we merely set \( g(z) = z + z^n/n \) and then \( J(z) = z + (a_n/n^2)z^n \). For many years, this conjecture was an open problem but in 1959 it was disproved by Löwner and Netanyahu [69]. However, the conjecture was proved to be true for various subclasses of \( S[41] \).

Asymptotic Bieberbach conjecture (1958). If \( A_n = \max_{f \in S} |a_n| \), then \( \lim_{n \to \infty} \frac{A_n}{n} = 1 \).
This weaker conjecture than the Bieberbach conjecture was proposed by Hayman [46] in 1958. In the same paper, he proved that $A_n/n$ tends to a limit. Nehari [76] proved that this conjecture implies the Littlewood conjecture.

Robertson second conjecture (1970). If $g(z) = \sum_{n=1}^{\infty} b_n z^n$ is quasi-subordinate to $f(z) = z + \sum_{n=1}^{\infty} a_n z^n$ where $f$ is in $S$, then $|b_n| \leq n$, $n \geq 1$.

The concept of quasi-subordination was introduced by Robertson [91,92] in 1970. We say that a function $g(z) = \sum_{n=1}^{\infty} b_n z^n$ is quasi-subordinate to $f(z) = \sum_{n=1}^{\infty} a_n z^n$ if both functions are analytic in $\Delta$ and there is a function $\Phi(z)$, analytic and bounded in $\Delta$ with $|\Phi(z)| \leq 1$, such that $g(z)/\Phi(z)$ is subordinate to $f(z)$ in $\Delta$.

The above conjecture is true for $n = 1$, $2$, $3$, $4$ and for all sufficiently large values of $n > n_0(f)$ [91]. It is true for all $n$ if the coefficients $a_n$ are all real ($b_n$ may be complex) [91]. It is true for all $n$ if $f$ is in $S^*$ or $H^\lambda$ or in $TR$ [91]. Further, it is observed by Robertson [92] in 1970 that the Robertson second conjecture (1970), if true, implies the truth of the Bieberbach conjecture and Rogosinski conjecture, and is true if the Robertson first conjecture (1936) is true. Thus all these conjectures are true if the Robertson first conjecture is true.

Milin conjecture (1971). For every $f \in S$,

$$\sum_{k=1}^{n} k(n+1-k)|c_k|^2 \leq \sum_{k=1}^{n} \frac{n+1-k}{k}, \quad n \geq 1$$

where $c_k$ are the logarithmic coefficients of $f \in S$ given by the equation

$$\log \frac{f(z)}{z} = 2 \sum_{k=1}^{\infty} c_k z^k, \quad z \in \Delta.$$

If $g(z) = \sum_{k=1}^{\infty} b_k z^{2k-1}$ is an odd power series such that $[g(z)]^2 = f(z^2)$, then

$$\sum_{k=1}^{\infty} b_k z^{2k-1} = b_1 \exp \left( \sum_{k=1}^{\infty} c_k z^k \right).$$
In 1967, Lebedev and Milin [75] obtained the inequality

$$\frac{1}{n+1} \sum_{k=1}^{n+1} |\beta_k|^2 \leq |\beta_1|^2 \exp\left\{ \frac{1}{n+1} \sum_{k=1}^{n} (n+1-k) \left( k |\sigma_k|^2 - \frac{1}{k} \right) \right\} .$$

Equality holds if and only if a complex number $w$ of absolute value one exists such that $\sigma_k = w^k/k$ for $k = 1, 2, \ldots, n$. Because of the Lebedev-Milin inequality (9), the Milin conjecture implies the Robertson conjecture and, so implies the Bieberbach conjecture [74]. Grinšpan 1972 [42] has verified the Milin conjecture up to $n = 3$.

Sheil-Small conjecture (1973). For each $f \in S$ and for each polynomial $P$ of degree $n$,

$$\|P \ast f\|_\infty \leq \|P\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the maximum modulus in $\Delta$, and $P \ast f$ stands for the convolution (or Hadamard product) of $P$ and $f$.

This conjecture of Sheil-Small [104] lies between the Robertson and Rogosinski conjectures. Taking $P(z) = z^n$, it can be easily seen that the Bieberbach conjecture is implied by the Sheil-Small conjecture.

Relationships between various conjectures

Interestingly, seven of the above mentioned conjectures are related as follows:

- Milin conjecture (1971) $\Longrightarrow$ Robertson conjecture (1936)
- Robertson conjecture (1936) $\Longrightarrow$ Sheil-Small conjecture (1973)
- Sheil-Small conjecture (1973) $\Longrightarrow$ Rogosinski conjecture (1943)
- Rogosinski conjecture (1943) $\Longrightarrow$ Bieberbach Conjecture (1916)
- Bieberbach Conjecture (1916) $\Longrightarrow$ Asymptotic Bieberbach conjecture (1958)
- Asymptotic Bieberbach conjecture (1958) $\Longrightarrow$ Littlewood conjecture (1925).

All these seven conjectures have remained open till as late as mid-summer 1984. In fact, the Bieberbach conjecture was considered so difficult to prove that some eminent mathematicians believed it to be false.
8. Discovery of the proof of Bieberbach conjecture

At last, the Bieberbach conjecture that has stumped several hundred mathematicians for about 70 years has now been solved by Louis de Branges. De Branges, 52, comes from Purdue University. He, in fact, has proved a stronger conjecture that was proposed by Milin in 1971. Thus he has settled all the seven conjectures as mentioned above. De Branges has claimed that he worked on the conjecture for 7 years before he had any success. He finally succeeded in May 1984. But the American mathematical community was not willing to read his 385 pages typed manuscript [59]. It was only in the Soviet Union that he finally got a hearing. Milin and his colleagues in Leningrad performed an extraordinary service to de Branges and mathematics, by being a patient audience. The proof has now been accepted by U.S. mathematicians too, and the results have appeared in the Acta Mathematica [18]. In October 1984, FitzGerald and Pommerenke [31] circulated an informal communication giving a shorter version of the de Branges theorem.

De Branges has proved the following result (Milin conjecture) which implies the Bieberbach conjecture via the Robertson conjecture.

De Branges Theorem [18]. Suppose that \( f \in S \) and write

\[
\log \frac{f(z)}{z} = 2 \sum_{k=1}^{\infty} c_k z^{k+1} \Delta (z). 
\]

Then, for \( n = 1, 2, ... \)

\[
\sum_{k=1}^{n} k(n+1-k)|c_k|^2 \leq \sum_{k=1}^{n} \frac{n+1-k}{k}. 
\]

(10)

Further, if \( f \in S \) and if

\[
f(z) \neq \frac{z}{(1-\bar{z}z)^2}, \quad |x| = 1
\]

then the strict inequality holds in (10).
The proof of de Branges is very ingenious in many respects. It makes use of an important result of Askey and Gasper [8] on Jacobi polynomials (namely, $\sum_{k=0}^{n} p_k^{(a,0)}(x) \geq 0$) that was published in 1976. The other result which was used is the Lebedev-Milin inequality (9).

The proof of the Milin conjecture given by de Branges depends on a continuous application of the Riemann mapping theorem which is due to Löwner [68]. Löwner used the method to prove the Bieberbach conjecture for the third coefficient. In this approach the problem is to propagate information by means of a differential equation. For this purpose, information has to be coded in a convenient form and then carried from one end of the interval to the other. However, the classical theories do not help because there is no fixed energy quadratic form which is preserved by the propagation. Thus de Branges had to develop some new techniques. An expository account of the new methods used by de Branges in proving this theorem are to appear in [19].

FitzGerald and Pommerenke [31] have shown that the de Branges method cannot work directly for the proof of the Bieberbach conjecture, and one has to use the ingenious Lebedev-Milin inequality (9) which enables one to prove first the Milin conjecture, leading to the solution of the Bieberbach conjecture via Robertson's proceedure.

FitzGerald and Pommerenke [31] have given a shorter version of the de Branges proof. The difference between these two proofs is purely technical. De Branges deduces his theorem by first proving a more general result on bounded univalent functions. He uses the ordinary Löwner differential equation which describes a contracting flow in the unit disk. On the other hand, FitzGerald and Pommerenke use the linear partial differential equation of Löwner which describes an expanding flow in the plane.

We close this section with a remark that a great deal of support work is generally necessary for any scientific breakthrough. More precisely, mathematics progresses by an accumulation of insights - including, of course, the major insights of Louis de Branges. Above, we have witnessed that the Milin conjecture was necessary for the proof of the Bieberbach conjecture. But the Milin conjecture would not have been made without a few fundamental
results, including the Robertson conjecture, the Grunsky inequalities, and Milin's earlier work showing $|a_n| \leq 1.243n$. Finally, de Branges has reduced his approach to the Bieberbach conjecture to an explicit question concerning special functions, specifically the non-negativity of certain sums related to Jacobi polynomials. That there would be any connection between univalent functions and these sums even now astonishes the mathematical community. These observations, indeed, speak of the interdisciplinary aspect of the de Branges proof.

9. What next after the Bieberbach conjecture?

The importance of the conjecture is mainly that it has proved so difficult and so much useful mathematics was developed as researchers tried to resolve it. However, it is too early to predict whether de Branges' method or his theorem will have any great significance for mathematics in general. We now discuss some of the related open problems as raised by earlier researchers. In view of the discovery of the proof of the Bieberbach conjecture and de Branges' method, one may now hope to solve some of these challenging problems.

Goodman conjecture for multivalent functions

A function $f$ is said to be a multivalent function of order $p$ (or $p$-valent function) in $\Delta$ if it assumes no value more than $p$ times in $\Delta$. We let $V(p)$ denote the class of all functions of the form $f(z) = b_1z + b_2z^2 + \ldots$, that are analytic and $p$-valent in $\Delta$. Goodman [38] in 1948 has made a conjecture analogous to the Bieberbach conjecture as follows:

If $f(z) = b_1z + \ldots + b_nz^n + \ldots$ belongs to $V(p)$, then

$$|b_n| \leq \sum_{k=1}^{p} \frac{2k(n+p)!}{(p+k)!(p-k)!(n-p-1)!(n^2-k^2)} |b_k|$$

(11)

for every $n \geq p + 1$.

For $p = 1 = b_1$, the Goodman conjecture yields the Bieberbach conjecture which now holds for the class $S$. The conjecture is known to be true for a few special subclasses of $V(p)$. Recently, Livingston [67] has shown that
(11) is true for all $n$ if $f(z)$ has the form $f(z) = b_{-1}z^{p-1} + b_1 z^p + \ldots$, $z \in \Delta$, and $f(z)$ is $p$-valently close-to-convex in $\Delta$. The reader will find in [39,40] some progress on the Goodman conjecture up to the year 1979.

Coefficient conjecture for class $\Sigma$

Closely related to the class $S$ is the class $\Sigma$ of functions $g(z) = z + b_0 + b_1/z + \ldots$ analytic and univalent in $|z| > 1$ except for a simple pole at $\infty$ with residue 1. The suggested conjecture for the class $\Sigma$ was $|b_n| \leq 2/(n+1)$, with equality for the function $z - 2z^{-n}/(n+1) + \ldots$. This conjecture has been proved to be true [24] for some special cases. However, the general conjecture is false even for the third coefficient. Bazilevič [10], in 1937, had shown that among all odd functions $g \in \Sigma$, the sharp upper bound for $|b_3|$ is $1/2 + e^{-6}$, and not $1/2$ as asserted by the conjecture. Thus the problem "Establish and prove a coefficient conjecture for the class $\Sigma"$ is still open. In fact, the coefficient problem for the class $\Sigma$ is considerably more difficult than the Bieberbach conjecture.

Extreme points of class $S$

A point $\eta$ in a convex set $A$ is called an extreme point of $A$ if it is not an interior point of any line segment contained in $A$. The closed convex hull of a set $A$ is the smallest closed convex set that contains $A$. Let $E(S)$ denote the set of all extreme points of $S$.

It is well known that $S$ is a compact subset of a locally convex topological space $U$ of all analytic functions in $\Delta$. By the Krein-Milman theorem [23], $S$ is contained in the closed convex hull of $S$. Thus the determination of $E(S)$ should provide tremendous information about $S$.

Open problem. Determine all the extreme points of $S$.

We observe that the Bieberbach conjecture for $E(S)$ is true because the conjecture is now known to be true for $S$. Further, the Koebe function and its rotations are certainly extreme points but $S$ has other extreme points as well. In fact, it is known [17] that closed convex hull of the Koebe function and its rotations contain $S^*$ but not all of $S$. Further, the extreme point theory has been applied to a number of problems involving
special subclasses of $S$ [71]. Hamilton [45] has now used quasiconformal mappings to investigate $E(S)$. However, the results of Hamilton are not known to the author. But the author feels that quasiconformal mappings can possibly be used to get $E(S)$.

General coefficient problem

In most general form, the coefficient problem is to determine the region $V_n$ of $\mathbb{C}^{n-1}$ occupied by the points $(a_2, a_3, \ldots, a_{n-1})$ for all $f \in S$. The special problem of estimating $|a_n|$ for all $f \in S$ is now being resolved by de Branges. But the general coefficient problem is still open and is really a challenging problem for analysts. A wealth of information about $V_n$ can be found in the book of Schaeffer and Spencer [97].

Coefficient problem for class $S_k$

Let $S_k$ be the class of homeomorphisms of the Riemann sphere which are analytic in $\Delta$ with normalization $f(0) = 0 = f'(0) - 1$ and which are $k$-quasiconformal in $\tilde{\Delta} = \{1 < |z| \leq \infty\}$. A function $f$ is said to be $k$-quasiconformal in $\tilde{\Delta}$ if relative to each standard rectangle in $\tilde{\Delta}$, $f$ is absolutely continuous on almost every horizontal and vertical line and satisfies the dilatation condition

$$1 \leq \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} \leq k \quad \text{a.e. in } \tilde{\Delta}.$$ 

Since 1-quasiconformality is conformality, $S_1$ consists of the Möbius mappings $z/(1-cz)$, $|c| \leq 1$. On the other extreme, as $k \to \infty$ the class $S_k$ becomes dense in the familiar class $S$. In 1976, Schiffer and Schober [101] raised the following coefficient problem for $S_k$:

Each $f \in S_k$ has the expansion $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in \Delta$.

Find $\max_{f \in S_k} |a_n|$.

Since $f \in S_k$ if and only if $e^{-i\alpha} f(e^{i\alpha} z) \in S_k$, $\alpha$ real, it is equivalent to find $\max_{f \in S_k} \text{Re } a_n$. 

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The problem of $\max_{f \in S_k} \Re \alpha_2$ is completely solved by Schiffer and Schober [101]. However the general problem, I believe, is still open. For some more recent open problems, a reader may refer to [107,108].

10. Conclusion

In this brief survey, I have made an attempt to document the importance of various methods generated or furthered by the Bieberbach conjecture. I have been compelled to omit a number of related interesting problems - open or solved. However, I have tried to convey that the Bieberbach conjecture has been an effective inspiration and testing ground for the developments in geometric function theory in the last seventy years.

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