CESARO SEQUENCE SPACES

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1. Introduction

The study of sequence spaces has been of great interest recently. A number of books have been published in this area over the last few years (see, for example, [6], [19] and [25]). In this lecture, we shall introduce two classes of sequence spaces, one of which is of a nonabsolute type and another of an absolute type. By absolute we mean that if a sequence \( x = \{x_k\} \) belongs to a given space, so does its absolute value \( |x| = \{|x_k|\} \). Otherwise the space is said to be nonabsolute. We shall characterize among others the dual spaces of these sequence spaces and give two applications in Sections 3 and 5 respectively.

For convenience, we list some of the well-known spaces as follows.

\[
\begin{align*}
c &= \text{the space of all convergent sequences,} \\
c_0 &= \text{the space of all null sequences,} \\
\ell_p &= \text{the space of all sequences } x = \{x_k\} \text{ such that} \\
          &\quad \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} < +\infty \text{ where } 1 \leq p < \infty, \\
\ell_\infty &= \text{the space of all bounded sequences,} \\
\ell_{bv} &= \text{the space of all sequences } x = \{x_k\} \text{ such that} \\
          &\quad \sum_{k=1}^{\infty} |x_k - x_{k+1}| < +\infty, \\
\ell_{bv_0} &= \ell_{bv} \cap c_0.
\end{align*}
\]

For suitable norms, the above are all complete normed linear spaces, i.e. Banach spaces [6].

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In what follows, we assume that the sequence spaces considered are normed and complete. Also, we write $x$ for $\{x_k\}$ and $y$ for $\{y_k\}$.

2. Cesaro sequence spaces of a nonabsolute type

Let $A$ be an infinite matrix. We define

$$\sigma_A = \{x; Ax \in \sigma\}.$$ 

This sequence space has been studied by many; see, for example, [24]. In general, let $Y$ be a given sequence space. We define

$$X = \{x; Ax \in Y\}.$$ 

In what follows, we assume that the mapping of $A$ from $X$ to $Y$ is one-one and onto. In particular, when $A$ is a Cesaro matrix $C$ where

$$C = \begin{pmatrix} 1 \\ 1/2 \\ 1/2 \\ 1/3 \\ 1/3 \\ 1/3 \\ \vdots \end{pmatrix}$$

and $Y = \ell_p$ for $1 \leq p \leq \infty$, we call $X$ Cesaro sequence space of a non-absolute type [17] and denote it by $X_p$. In other words, $x \in X_p$ for $1 \leq p < \infty$ if and only if

$$\left\{ \sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right|^p \right\}^{1/p} < +\infty,$$

and similarly for $p = \infty$.

This class of sequence spaces has some undesirable properties. First, it does not necessarily contain all finite sequences. For example, $e^1 = (1,0,0,...) \notin X_1$; however $X_1$ is not empty since $\{(k+1)^{-1} - k^{-1}\}_{k \geq 1} \in X_1$. Secondly, it is not solid. A set $X$ is said to be solid if $x \in X$ and $|y| \leq |x|$ imply that $y \in X$. Note that if $X$
is solid then $X$ is also absolute. Now let $x = \{x_k\}$ and $x_k = (-1)^k$. Then $x \in X_\infty$ but $|x| \notin X_\infty$ for $1 < p < \infty$. That is, $X_\infty$ is nonabsolute for $1 < p < \infty$, indeed also for $p = 1$ and $\infty$, therefore $X_\infty$ cannot be solid. We shall see that many results remain valid without the above two properties which are usually assumed in the study of sequence spaces.

One area of study in sequence spaces is matrix transformations. Let $T = (t_{nk})$ be an infinite matrix mapping a sequence space $X$ into another sequence space $Z$. That is, $y = Tx \in Z$ whenever $x \in X$ where

$$y_n = \sum_{k=1}^{\infty} t_{nk} x_k .$$

The problem is to find necessary and sufficient conditions on $T$ such that $T$ maps $X$ into $Z$. A necessary condition is that for each $n$ and each $x \in X$ the series

$$\sum_{k=1}^{\infty} t_{nk} x_k$$

must converge.

In other words, $\{t_{nk}\}_{k \geq 1}$ belongs to the $\beta$-dual of $X$ for each $n$. We define two kinds of dual spaces here. The $\beta$-dual of $X$, denoted by $X^\beta$, is the space of all sequences $y$ such that for every $x \in X$

$$\sum_{k=1}^{\infty} x_k y_k$$

converges.

The $\alpha$-dual of $X$, denoted by $X^\alpha$, is the space of all sequences $y$ such that for every $x \in X$

$$\sum_{k=1}^{\infty} |x_k y_k|$$

converges.

If the space $X$ is solid, then $X^\alpha = X^\beta$. For example, $l_\infty^\alpha = l_\infty^\beta = l_1$ where $1 \leq p \leq \infty$ and $1/p + 1/q = 1$. However $bv^\alpha = l_1$ whereas $bv^\beta = cs$. Here $cs$ denotes the space of all sequences $x$ such that
In order to study matrix transformations on $X_p$, the first natural question to ask is what is the $\beta$-dual of $X_p$.

Suppose $y \in X_p^\beta$ where $1 < p < \infty$. Consider

$$
\sum_{k=1}^{n} x_k y_k = \sum_{k=1}^{n-1} \left( \frac{1}{k} \sum_{i=1}^{k} x_i \right) (k(y_k - y_{k+1})) + \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right) (ny_n)
$$

$$
= \sum_{k=1}^{n} b_{nk} s_k
$$

where

$$
b_{nk} = \begin{cases} 
  k(y_k - y_{k+1}) & \text{when } 1 \leq k \leq n-1, \\
  ny_n & \text{when } k = n, \\
  0 & \text{when } k > n
\end{cases}
$$

and $s_k = \frac{1}{k} \sum_{i=1}^{k} x_i$.

Let $B = (b_{nk})$. Then for every $x \in X_p$ and $s = \{s_k\} \in \ell_p$, we have $Bs \in c$. It is well-known [13] that $B$ maps $\ell_p$ into $c$ if and only if

$$
\lim_{n \to \infty} b_{nk} \text{ exists for each } k , \text{ and}
$$

$$
\sup \left\{ \left( \sum_{k=1}^{n} |b_{nk}|^q \right)^{1/q}; n \geq 1 \right\} < +\infty
$$

where $1/p + 1/q = 1$. Note that the first condition is trivially satisfied. The second condition gives the following [17].
Theorem 1. The $\beta$-dual of $X_p$ for $1 < p < \infty$ is the space of all sequences $y$ such that

$$\sup \left\{ \left( \sum_{k=1}^{n-1} |k(y_k - y_{k+1})|^q + |ny_n|^q \right)^{1/q} ; \ n \geq 1 \right\} < +\infty .$$

We remark that similar results hold for $p = 1$ and $\infty$.

Alternatively, we may split the condition into two as follows:

(1) $\sup \{ |ny_n| ; \ n \geq 1 \} < +\infty$;

(2) $\left( \sum_{k=1}^{\infty} |k(y_k - y_{k+1})|^q \right)^{1/q} < +\infty$.

Using condition (1), we see that

$$\sum_{k=1}^{\infty} x_k y_k = \sum_{k=1}^{\infty} s_k t_k$$

where $s_k$ is defined as above and $t_k = k(y_k - y_{k+1})$. In other words, the left hand side converges if and only if the right hand side also converges. Since it is easy to show that condition (1) is necessary, we have an alternative proof of Theorem 1.

In fact, the first proof is rather general. We can easily extend it to a general case. For a given sequence $x$, we write

$$x^N = (x_1, x_2, \ldots, x_N, 0, \ldots).$$

Then we say a sequence space $X$ has the $AK$ property if for each $x \in X$ we have $x^N \in X$ and $\|x^N - x\| \to 0$ as $N \to \infty$. Further, given a norm $\| \cdot \|$ in $X$, we define an associate norm as follows:

$$\|y\|^* = \sup \{ \sum_{k=1}^{\infty} |x_k y_k| ; \|x\| \leq 1 \} .$$

Hence we can prove [9]
Theorem 2. Let \( A \) be an infinite matrix which is one-one and onto and \( Y \) a sequence space having the AK property. Let \( X = \{ x ; Ax \in Y \} \). Then the \( \beta \)-dual of \( X \) is the space of all sequences \( y \) such that \( yA^{-1} \) exists and

\[
\sup \{ \| y^N A^{-1} \|_Y ; N \geq 1 \} < + \infty .
\]

Obviously, Theorem 1 is a special case of Theorem 2. For other special cases, see, for example, [16]. We remark that given a sequence space \( X \) which is nonabsolute, an appropriate norm in \( X^\beta \) is given by

\[
\| y \|_X^\beta = \sup \{ \| y^N \|_X^* ; N \geq 1 \}
\]

and not \( \| y \|_X^* \). The latter is suitable only for sequence spaces that are absolute.

3. Matrix transformations

We state a theorem due to Zeller.

Theorem 3. Let \( X, Y \) and \( Z \) be sequence spaces with \( Y \) having the AK property and \( A \) an infinite matrix such that \( A : X \to Y \) is one-one and onto. Then \( T = (t_{nk}) \) maps \( X \) into \( Z \) if and only if

(i) \( \{ t_{nk} \}_k \geq 1 \in X^\beta \) for each \( n \),

(ii) \( TA^{-1} \) maps \( Y \) into \( Z \).

In view of Theorem 2, we can replace condition (i) above by those given in Theorem 2. In particular, we have

Theorem 4. An infinite matrix \( T = (t_{nk}) \) maps \( X_p \) into \( c \) for \( 1 < p < \infty \) if and only if
(i) \( \sup\{|kt_{nk}|; k \geq 1\} < +\infty \) for each \( n \),

(ii) \( \sum_{k=1}^{\infty} |k(t_{nk} - t_{n,k+1})|^g < +\infty \) for each \( n \),

(iii) \( \lim_{n \to \infty} (t_{nk} - t_{n,k+1}) \) exists for each \( k \),

(iv) \( \sup\{ \sum_{k=1}^{n-1} |k(t_{nk} - t_{n,k+1})|^g + |nt_{mn}|^g; n \geq 1\} < +\infty \).

Similarly, we can state the result for \( p = 1 \) and \( \infty \). Note that as long as we can characterize \( TC^{-1}: l_p \to Z \), which is normally easier, then we can characterize \( T: X_p \to Z \). Hence we may obtain many other results.

The proof of Theorem 3 depends on the following associative property: \( T(A^{-1}y) = (TA^{-1})y \) for all \( y \in Y \), which follows from the \( AK \) property of \( Y \) by a theorem of Zeller [26; Theorem 17. vii]. We remark that Jakimovski and Livne [5] proved a similar result. There they assume that \( Y \) satisfies a more general condition than \( AK \).

In the same way, we can characterize \( T: Z \to X \) where \( Z \) has the \( AK \) property and \( X \) is defined as in Theorem 3. Other extensions are indicated in [9]. For a list of results on the characterization of infinite matrices from some well-known sequence spaces into others, see [22].

4. Cesaro sequence spaces of an absolute type

Now we consider the absolute version. Let \( A \) be an infinite matrix and \( Y \) a sequence space. We consider

\[ X = \{x; A|x| \in Y\}. \]

In particular, when \( A \) is a Cesaro matrix \( C \) and \( Y = l_p \) for \( 1 < p < \infty \), we call \( X \) Cesaro sequence space of an absolute type and denote it by
ces_p. In other words, \( x \in ces_p \) for \( 1 < p < \infty \) if and only if

\[
\left[ \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right]^{1/p} < +\infty,
\]

and similarly for \( p = \infty \). We do not consider the case when \( p = 1 \) because the space is trivial. The space \( ces_p \) was first defined by Shiue [20], its \( \alpha \)-dual given by Jagers [4] for \( 1 < p < \infty \), and by Ng and Lee [15] for \( p = \infty \). The version we shall present here is due to Lim [12].

The first problem is to find the \( \alpha \)-dual of \( ces_p \). We remark that the proof that works for the nonabsolute case does not go through here. We observe that

\[
\sum_{k=1}^{n} |x_k y_k| = \sum_{k=1}^{n-1} \left( \frac{1}{k} \sum_{i=1}^{n} |x_i| \right) (k(|y_k| - |y_{k+1}|)) + \left( \frac{1}{n} \sum_{i=1}^{n} |x_i| \right) (n|y_n|)
\]

\[
= \sum_{k=1}^{n-1} s_k t_k + \text{tail}
\]

where \( s_k = \frac{1}{k} \sum_{i=1}^{k} |x_i| \) and \( t_k = k(|y_k| - |y_{k+1}|) \).

If the tail tends to zero and \( t = \{t_k\} \in l_q \) with \( 1/p + 1/q = 1 \), then \( y \in ces_p \). But the converse does not hold because the series

\[
\sum_{k=1}^{\infty} |x_k y_k| = \sum_{k=1}^{\infty} s_k t_k
\]

if exists, converges for some \( s = \{s_k\} \in l_p \) only. Hence a different approach is required.

Suppose \( y \in ces_p \). Using functional analytic method [26; Theorem 17. iii], we can show that for every \( x \in ces_p \)

\[
\sum_{k=1}^{\infty} |x_k y_k| \leq \mathcal{N} \|x\|.
\]
Here the norm is defined to be
\[ \| x \| = \left[ \sum_{k=1}^{\infty} \left( \frac{1}{k} \sum_{i=1}^{k} |x_i| \right)^p \right]^{1/p} \]

It is easy to show that \( y \in c_0 \) and the tail as given above tends to zero. Then we obtain
\[ \sum_{k=1}^{\infty} \left( \frac{1}{k} \sum_{i=1}^{k} |x_i| \right) (k(|y_k| - |y_{k+1}|)) \leq M \left[ \sum_{k=1}^{\infty} \left( \frac{1}{k} \sum_{i=1}^{k} |x_i| \right)^p \right]^{1/p} \]
or alternatively,
\[ \sum_{k=1}^{\infty} s_k t_k \leq M \left( \sum_{k=1}^{\infty} s_k^p \right)^{1/p} \]

In order to say \( t = \{ t_k \} \in \ell_q \), we require \( s_k = t_k^{q-1} \). In other words, we want to choose \( x \in \text{ces}_P \) such that
\[ \frac{1}{k} \sum_{i=1}^{k} |x_i| = (k(|y_k| - |y_{k+1}|))^{q-1} \]
or equivalently,
\[ \sum_{i=1}^{k} |x_i| = k^q (|y_k| - |y_{k+1}|)^{q-1} \]
\[ = \left[ k^p (|y_k| - |y_{k+1}|) \right]^{q-1} \]

This is possible only when \( k^p (|y_k| - |y_{k+1}|) \) is non-negative and increasing in \( k \). Therefore we have proved

Lemma 5. The space \( \text{ces}_P^\alpha \) contains all \( y \) such that \( y \in c_0 \) and
\[ \sum_{k=1}^{\infty} \left[ k(|y_k| - |y_{k+1}|) \right]^q < +\infty \]

where \( k^p(|y_k| - |y_{k+1}|) \) is non-negative and increasing; and its solid hull.

Rewriting the above lemma, we have

**Theorem 6.** The a-dual of \( \text{ces}_p \) for \( 1 < p < \infty \) is the space of all sequences \( y \) such that \( y \in c_0 \) and

\[ \sum_{k=1}^{\infty} |k(y_k - y_{k+1})|^q < +\infty \]

where \( y^* \) denotes the infimum of all \( y^* \leq |y| \) with \( k^p(y_k^* - y_{k+1}^*) \) being non-negative and increasing.

Again, let \( X = \{ x; A|x| \in Y \} \). Lim [12] considered the cases for other special \( A \) and \( Y \); in particular, when \( A = C \), Cesaro matrix, and \( Y = w_0 \) where \( w_0 = \{ x; C|x| \in c_0 \} \).

5. Cesaro function spaces

There is a function version of Cesaro sequence spaces [21]. Cesaro function space, denoted by \( CES_p \) for \( 1 < p < \infty \), is defined to be the set of all real-valued measurable functions defined on \((0, \infty)\) such that

\[ \left[ \int_0^\infty \left( \frac{1}{x} \int_0^x |f(t)| dt \right)^p dx \right]^{1/p} \]

converges.

In other words, \( f \in CES_p \) if and only if \( T|f| \in L_p(0,\infty) \) where

\[ (T|f|)(x) = \frac{1}{x} \int_0^x |f(t)| dt . \]
Obviously, the operator $T$ plays the role of matrix $A$ or $C$ as in the sequence version.

Given a function space $X$ with functions defined on $(0,\infty)$, the Kothe dual of $X$ is the space of all real-valued measurable functions $g$ defined on $(0,\infty)$ such that

$$
\int_0^\infty f(x)g(x)\,dx \text{ exists for every } f \in X .
$$

Unfortunately, it is difficult to express the inverse of $T$ precisely. Hence the technique used in the sequence version to find the dual does not work here. However it is still possible to find the dual of $CES_\mathbb{P}$. We do it by first of all converting the problem in function spaces to one in sequence spaces, and then solve the problem in sequence spaces.

We observe that for $n \leq x \leq n+1$

$$
\frac{1}{2n} \int_0^n \leq \frac{1}{n+1} \int_0^x |f(t)|\,dt \leq \frac{1}{n} \int_0^{n+1} \leq \frac{2}{n+1} \int_0^{n+1}
$$

In other words, the integral

$$
\int_1^\infty \left( \frac{1}{x} \int_0^x |f(t)|\,dt \right)^p \, dx
$$

converges if and only if the series

$$
\sum_{n=1}^\infty \left( \frac{1}{n} \sum_{k=1}^n \right)^p
$$

converges

where

$$
e_k = \int_{k-1}^k |f(t)| \, dt .
$$

That is, $s = \{e_k\} \in ces_\mathbb{P}$.
Next, we write for \((m+1)^{-1} \leq x \leq m^{-1}\)

\[
\frac{m+1}{2} \int_0^{1/(m+1)} \leq m \int_0^{1/(m+1)} \leq \frac{1}{x} \int_0^x |f(t)| dt \leq (m+1) \int_0^{1/m} \leq 2m \int_0^{1/m} .
\]

In other words, the integral

\[
\int_0^1 \left( \frac{1}{x} \int_0^x |f(t)| dt \right)^p dx
\]

converges if and only if the series

\[
\sum_{m=1}^{\infty} \left( \sum_{k=m}^{\infty} t_k \right)^p \left( \frac{1}{m} - \frac{1}{m+1} \right)
\]

converges where

\[
t_k = \int_{1/k}^{1/(k+1)} |f(t)| dt .
\]

We say that \(t = \{t_k\}\) belongs to a reverse Cesaro sequence space or \(t \in d_p\).

Combining the above, we obtain that \(f \in CES_p\) where \(1 < p < \infty\) if and only if \(s = \{s_k\} \in ces_p\) and \(t = \{t_k\} \in d_p\) where

\[
s_k = \int_{k-1}^k |f(t)| dt \quad \text{and} \quad t_k = \int_{1/k}^{1/(k+1)} |f(t)| dt .
\]

Since the \(\alpha\)-dual of \(ces_p\) is known and that of \(d_p\) can be found in a similar way [23], therefore we have solved the problem of the Kothe dual of \(CES_p\).

**Theorem 7.** The Kothe dual of \(CES_p\) for \(1 < p < \infty\) is the space of all real-valued measurable functions \(g\) such that \(u = \{u_k\} \in ces_p\) and
\[ v = \{ v_k \} \in d^\alpha_p \text{ where} \]
\[ u_k = \text{ess-sup} \{ |g(x)|; k-1 \leq x \leq k \}, \]
\[ v_k = \text{ess-sup} \{ |g(x)|; (k+1)^{-1} \leq x \leq k^{-1} \}. \]

6. Some problems

We list in the following some problems which are being studied by my students and some Chinese mathematicians in Shanghai and Guangzhou.

Problem 1. **What is the Banach dual of** \( X_p \) **?**

For an absolute sequence space, very often its Banach dual (i.e. the space of all continuous linear functionals) coincides with the \( \alpha \)-dual, and also with the \( \beta \)-dual. But the space \( X_p \) is nonabsolute and its Banach dual includes the \( \beta \)-dual as a proper subspace.

Problem 2. **What is the second \( \beta \)-dual of** \( X_p \) **?**

We have characterised the \( \beta \)-dual \( X^\beta_p \). The conditions on the elements in \( X^\beta_p \) are not symmetrical with respect to those in \( X_p \). Hence the space \( X_p \) is not perfect, i.e. \( X^\beta \beta_p \neq X_p \). Then what is \( X^\beta \beta_p \)?

Some results on the above two problems have been obtained by Ou Zeming and Wu Bo-er.

Problem 3. **Let** \( X = \{ x; A|x| \in Y \} \) **and** \( Y \) **be solid. What is the** \( \alpha \)-**dual of** \( X \) **?**

Lim [12] has solved the problem for some special \( A \) and \( Y \). The corresponding problem has also been solved completely for the nonabsolute case. It would be of interest to characterize the \( \alpha \)-dual of \( X \) for more general \( A \) and \( Y \). If \( Y \) is not solid, the problem seems to be harder.
Problem 4. Let $X$ be a Banach space. Study the space $c_{es_p}(X)$ of all $X$-valued sequences $x$ such that
\[ \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} \|x_k\|^p \right)^{\frac{1}{p}} < +\infty, \]
and similarly for the nonabsolute case.

The problem was proposed and is being studied by Wu Bo-er. For similar results on $c_0(X)$, $c(X)$, $l_p(X)$ for $0 < p < \infty$, and $\omega_p(X)$ for $0 < p < \infty$, see [14].

Problem 5. Let $T$ be a family of infinite matrices and $Y$ a sequence space. Let $X$ be the space of all sequences $x$ such that $Tx \in Y$ whenever $T \in T$ and
\[ \sup \{ \|Tx\|; T \in T \} < +\infty. \]

What is the $\beta$-dual of $X$?

The above space $X$ was studied by Lim and Lee [11].

Problem 6. Let $\omega_p$ denote the space of all sequences $x$ such that there is a number $\lambda$ satisfying
\[ \frac{1}{n} \sum_{k=1}^{n} |x_k - \lambda|^p \to 0 \quad \text{as} \quad n \to \infty. \]

What are the necessary and sufficient conditions on $A$ such that an infinite matrix $A$ maps $\omega_1$ into $\omega_p$ for $1 < p < \infty$?

This is a special case of a problem raised in Kuttner and Thorpe [8]. An attempt has been made in Lee and Lim [10].

Problem 7. Characterize the continuous orthogonally additive functionals on $X_p$. 

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Given a sequence space $X$, a functional $f$ on $X$ is said to be orthogonally additive if

$$f(x + y) = f(x) + f(y)$$

whenever $x, y \in X$ and $x_k y_k = 0$ for all $k$. Chew [1] has characterized the continuous orthogonally additive functionals on $\ell_p$, Orlicz sequence spaces and a few others. It seems that the nonabsolute case is much harder than the following absolute case.

Problem 8. Characterize the continuous orthogonally additive functionals on $\text{ces}_p$.

The above list, except Problem 4, coincides with that given at the end of a lecture series in Shanghai in 1983. The original Problem 4 was to find the $\alpha$-dual of $d_p$ and was solved by Zhang [23]. The geometry of these spaces is also being studied by Yu Xintai and Zhang. For interest, we include in the references some standard reference books on the subject: [2], [3], [7], [18] and [26].

REFERENCES


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