INTIAL VALUE PROBLEMS FOR SINGULARLY PERTURBED NON LINEAR ORDINARY DIFFERENTIAL EQUATIONS.*

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1. Introduction.

By way of introduction we consider the following class of so-called singular perturbation problems:

\[ \varepsilon L_2[u_\varepsilon] + L_1[u_\varepsilon] = f(x), \quad x \in \Omega \subset \mathbb{R}^n, \quad (1.1) \]

with boundary conditions for \( u_\varepsilon \). \( L_1 \) and \( L_2 \) are differential operators with order of \( L_2 \) larger than that of \( L_1 \) and \( \varepsilon \) is a small positive parameter, appearing only as a factor of \( L_2 \). Instead of (1.1) we consider the reduced boundary value problem \((\varepsilon = 0)\):

\[ L_1[w] = f(x), \quad x \in \Omega \subset \mathbb{R}^n, \quad (1.2) \]

with \( w \) submitted to certain boundary conditions to be chosen from those applied to \( u_\varepsilon \). One can submit \( w \) only to part of the boundary conditions for \( u_\varepsilon \), because the order of \( L_1 \) is smaller than that of \( L_2 \). The key question in singular perturbation theory is the following:

When does \( \lim_{\varepsilon \to 0} u_\varepsilon = w \)? \( (1.3) \)

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Whenever this question is answered positively, the concept of limit has
to be specified in terms of a suitable norm. The next question is: how
good is the approximation $w$ for the unknown solution $u_\varepsilon$? In other
words, is it possible to specify a norm $\| \cdot \|$ and a positive number $\nu$, such that

$$\| u_\varepsilon - w \| = O(\varepsilon^\nu)$$  \hfill (1.4)

There exists an extensive literature on this subject and many cases have
been considered as to the type of the operators $L_1$ and $L_2$. For
example $L_1$ may be of the first order and $L_2$ may be of elliptic,
parabolic or hyperbolic type; also other cases are possible such as $L_1$
and $L_2$ are both of elliptic type. We refer the reader to references
[1]-[13]. The case of singular perturbations in which $L_2$ is of hyperbolic type has been investigated satisfactorily only rather recently.
We refer to the works of Genet-Madaune [12], Weinstein-Smith [13] and
that of the present authors [10] and [11].

In all investigations concerning the answer to the questions (1.3)
and (1.4), a priori estimates for solutions of partial differential
equations play an important role. For singular perturbations of hyperbolic type the authors have emphasized the use of energy integrals for
making the necessary a priori estimates (see [10] and [11]).

Because the method for solving the singular perturbation problem

$$\varepsilon \left\{ \frac{\partial^2 u}{\partial t^2} - c^2(x,t) \frac{\partial^2 u}{\partial x^2} \right\} + a(x,t,u) \frac{\partial u}{\partial x} + b(x,t,u) \frac{\partial u}{\partial t} + d(x,t,u) = 0$$

$$-\infty < x < +\infty, \ 0 < t < \infty$$  \hfill (1.5)

with the initial conditions

$$u(x,0) = f(x), \ \frac{\partial u}{\partial t}(x,0) = g(x), \ -\infty < x < +\infty$$  \hfill (1.6)
may be obtained by a straightforward but technically rather complicated generalization of the method for solving the singular perturbation problem

\[ \varepsilon \frac{d^2 u}{dt^2} + a(t,u) \frac{du}{dt} + b(t,u) = 0, \quad 0 < t < \infty \quad (1.7) \]

\[ u(0) = \alpha, \quad \frac{du}{dt}(0) = \beta, \quad (1.8) \]

we restrict ourselves in this paper to the latter initial value problem which involves only an ordinary differential equation, but is of enough mathematical interest in itself. For the generalization to the Cauchy problem (1.5)-(1.6), we refer the reader to [11]. We assume the coefficients \( a \) and \( b \) to belong to the class \( C^3(\mathbb{R}_+ \times \mathbb{R}) \) and moreover the coefficient \( a \) has to satisfy the condition

\[ a(t,u) \geq a_0 > 0, \quad \forall t \in \mathbb{R}_+, \quad \forall u \in \mathbb{R}, \quad (1.9) \]

where \( a_0 \) is an arbitrary positive number. The reduced problem reads as

\[ a(t,\omega) \frac{d\omega}{dt} + b(t,\omega) = 0 \quad (1.10) \]

with

\[ \omega(0) = \alpha. \quad (1.11) \]

The question to be solved is: does there exist a suitable norm and a positive number \( \nu \) such that

\[ \|u - \omega\| = O(\varepsilon^{\nu})? \quad (1.12) \]
2. The Formal Approximation.

It is clear that the solution \( \psi \) of the reduced problem (1.10)–(1.11) does not satisfy in general the second initial condition (1.8). In order to comply with this condition the coordinate \( t \) is stretched as \( \tau = \frac{t}{\varepsilon} \) and we consider as an approximation for the unknown solution \( \psi \) of (1.7)–(1.8) the function

\[
\tilde{u}(t) = \psi(t) + \varepsilon \nu\left(\frac{t}{\varepsilon}\right). \tag{2.1}
\]

Inserting (2.1) into (1.7) and returning only the zero order term in \( \varepsilon \), we get for the correction function \( \nu(\tau) \) the linear differential equation

\[
\frac{d^2
u}{d\tau^2} + a(0,\psi(0))\frac{d
u}{d\tau} = 0. \tag{2.2}
\]

The requirement that \( \tilde{u} \) should satisfy the second initial condition (1.8) yields

\[
\left. \frac{d\psi}{dt} \right|_{t=0} = \psi'(0) + \left. \frac{d\nu}{d\tau} \right|_{\tau=0} = \beta \tag{2.3}
\]

where a dash denotes differentiation with respect to \( t \). A second condition for the function \( \nu(\tau) \) is introduced by the option that the correction \( \nu \) should have only a significant influence in an \( \varepsilon \)-neighbourhood of \( t = 0 \), this means

\[
\lim_{\tau \to \infty} \nu(\tau) = 0. \tag{2.4}
\]

It follows now from (2.2), (2.3) and (2.4) that \( \nu\left(\frac{t}{\varepsilon}\right) \) is given by the expression

\[
\nu\left(\frac{t}{\varepsilon}\right) = \frac{\psi'(0) - \beta}{a(0,\alpha)} \exp \left[ -a(0,\alpha) \frac{t}{\varepsilon} \right], \tag{2.5}
\]
which, due to (1.9), is asymptotically zero for \( t \geq \delta > 0 \), with \( \delta \) any positive number, independent of \( \epsilon \). The term \( \epsilon u(\frac{t}{\epsilon}) \) in (2.1) has a typical boundary layer character. The solution \( u(t) \) of the reduced problem (1.10)-(1.11), which is nonlinear, may have singularities.

From now on we assume \( u(t) \) twice continuously differentiable for \( 0 \leq t \leq T \), where \( T \) is some positive number; moreover we consider \( u(t) \) as a known function in this segment. Inserting (2.1) into (1.7) and (1.8) we obtain that \( \tilde{u} \) satisfies the initial value problem

\[
\frac{\epsilon^2 d^2 \tilde{u}}{dt^2} + a(t, \tilde{u}) \frac{d \tilde{u}}{dt} + b(t, \tilde{u}) = 0(\epsilon), \quad \text{uniformly in } [0,T] \tag{2.6}
\]

with

\[
\tilde{u}(0) = \alpha + \epsilon v(0) = \alpha + O(\epsilon) \tag{2.7}
\]

and

\[
\tilde{u}'(0) = \beta . \tag{2.8}
\]

Hence \( \tilde{u} \) satisfies the original initial value problem (1.7)-(1.8) up to \( O(\epsilon) \) in the segment \([0,T]\) and so \( \tilde{u} \) is a good candidate for being a good approximation for the solution \( u \) of the initial value problem (1.7)-(1.8) for values of \( t \) in the segment \([0,T]\). In order to verify this guess we introduce the remainder term \( R \), defined by

\[
u(t) = \tilde{u}(t) + R(t), \tag{2.9}
\]

and \( R(t) \) has to be estimated in \([0,T]\). From (1.7)-(1.8) and (2.6)-(2.8) it follows that \( R \) satisfies the initial value problem:

\[
\epsilon R'' + a(t, \tilde{u} + R) R' + \{ a(t, \tilde{u} + R) - a(t, \tilde{u}) \} \tilde{u}' + \{ b(t, \tilde{u} + R) - b(t, \tilde{u}) \} = O(\epsilon),
\]

uniformly in \([0,T]\), with
Finally, putting
\[ R(0) = -\varepsilon v(0) \quad \text{and} \quad R'(0) = 0 . \]
we get for \( \overline{R} \) the initial value problem
\[ F(\overline{R}) = \varepsilon \overline{R}'' + a(t,\overline{u}+\overline{R})\overline{R}' + \{a(t,\overline{u}) - a(t,\overline{u})\} \overline{u}' + \{b(t,\overline{u}+\overline{R}) - b(t,\overline{u})\} = 0(\varepsilon) , \]
uniformly in \([0,T]\) with
\[ \overline{R}(0) = \overline{R}'(0) = 0 . \]

The function \( \overline{R}(t) \), and hence also the remainder term \( R(t) \), will now be estimated in the segment \([0,T]\) from the initial value problem (2.11)-(2.12). In order to do this we shall use a fixed point theorem, introduced by Van Harten [7] for estimating remainder terms arising in the theory of singular perturbations of elliptic type. We use here a modified version of this theorem.

3. Fixed Point Theorem

Let \( N \) be a normed linear space, consisting of elements \( p \) with norm denoted by \( |p| \), and \( B \) be a Banach space, consisting of elements \( q \) with norm denoted by \( \|q\| \). Let \( F \) be a nonlinear map \( N \rightarrow B \) with \( F(0) = 0 \) and assume that \( F(p) \) can be decomposed as
\[ F(p) = L(p) + \psi(p) , \]
with \( L \) the linearization of \( F \) at \( p = 0 \). Finally the operators \( L \) and \( \psi \) are submitted to the following conditions:

(1) \( L \) is bijective and its inverse \( L^{-1} \) is continuous, i.e.
\[ |L^{-1}[q]| \leq \ell^{-1}\|q\|, \quad \forall q \in B , \]

with \( \ell \) the linearization of \( F \) at \( p = 0 \). Finally the operators \( L \) and \( \psi \) are submitted to the following conditions:
where \( \ell \) is some positive number.

(2) Let \( \Omega_N(\rho) \) be the ball \( \{p \mid p \in N, |p| \leq \rho\} \). There exists a number \( \tilde{\rho} > 0 \) such that

\[
\|\psi(p_1) - \psi(p_2)\| \leq m(\rho)|p_1 - p_2|, \quad \forall p_1, p_2 \in \Omega_N(\rho) \text{ with } 0 \leq \rho \leq \tilde{\rho}
\]

where \( m(\rho) \) is monotonically decreasing for \( \rho \to 0 \) with

\[
\lim_{\rho \to 0} m(\rho) = 0.
\]

Theorem. If \( \rho_0 = \sup_{\rho \geq 0} \{\rho \mid 0 < \rho \leq \tilde{\rho}, m(\rho) \leq \frac{1}{2}\ell\} \), then there exists for any \( f \in B \) with \( \|f\| \leq \frac{1}{2}\ell\rho_0 \) an element \( p \in N \) such that \( F(p) = f \) with

\[
|p| \leq 2\ell^{-1}\|f\|.
\]

Proof. Due to the bijectivity of the map \( L(p) = q \), the equation

\[
F(p) = f, \quad p \in \Omega_N(\tilde{\rho}), \quad f \in B
\]

is equivalent to the equation

\[
q = f - \psi o L^{-1}[q], \quad q \in L[\Omega_N(\tilde{\rho})], \quad f \in B.
\]

We consider now the ball \( \Omega_B(\ell\rho) \subset B \) with centre 0 and radius \( \ell\rho \), with \( 2\ell^{-1}\|f\| \leq \rho \leq \rho_0 \leq \tilde{\rho} \).

The operator \( T \) has the following properties:

(a) \( T \) is defined in the ball \( \Omega_B(\ell\rho) \).

For any \( q \in \Omega_B(\ell\rho) \) we have \( q = L o L^{-1}[q] \), with

\[
|L^{-1}[q]| \leq \ell^{-1}\|q\| \leq \rho \leq \rho_0 \leq \tilde{\rho}.
\]

Hence \( \Omega_B(\ell\rho) \subset L[\Omega_N(\tilde{\rho})] \).

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Since $T$ is defined in $L[N_\rho(p)]$, it is also defined in $\Omega_B(\ell\rho)$.

(b) $T$ is strictly contractive on $\Omega_B(\ell\rho)$.

For any $q_1, q_2 \in \Omega_B(\ell\rho)$ we have the inequalities:
$$\|T[q_1] - T[q_2]\| = \|\psi \circ L^{-1}[q_1] - \psi \circ L^{-1}[q_2]\|$$
$$\leq m(\rho) |L^{-1}[q_1] - L^{-1}[q_2]| \leq m(\rho) \ell^{-1} \|q_1 - q_2\| \leq \frac{1}{2}\|q_1 - q_2\| .$$

(c) $T[\Omega_B(\ell\rho)] \subset \Omega_B(\ell\rho)$.

For any $q \in \Omega_B(\ell\rho)$ we have the inequalities:
$$\|T[q]\| \leq \|f\| + \|\psi \circ L^{-1}[q]\| \leq \frac{1}{2}\ell\rho + m(\rho) |L^{-1}[q]|$$
$$\leq \frac{1}{2}\ell\rho + m(\rho) \rho \leq \ell\rho .$$

Hence there exists a unique solution of equation (3.6) in the ball $\Omega_B(\ell\rho)$ and due to the equivalence of (3.6) and (3.5) there exists also a solution $\rho \in \Omega_N(\rho)$ of equation (3.5). (Note that uniqueness of $\rho$ has not been established in the whole ball $\Omega_N(\rho)$, but only in $L^{-1}[\Omega_B(\ell\rho)] \subset \Omega_N(\rho)$).

Further we have
$$|p| = |L^{-1}[q]| \leq \ell^{-1}\|q\| \leq \rho$$
Choosing finally $\rho = 2\ell^{-1}\|f\|$, we get the estimate (3.4).

4. The Estimation of $R$

In order to apply the above fixed point theorem to the estimation of $R$ we consider the non linear map $F$, specified in (2.11). This map may be decomposed as
$$F(p) = L(p) + \psi(p)$$
(4.1)
with

\[ L(p) = \varepsilon \frac{d^2p}{dt^2} + a(t, \overline{u}) \frac{dp}{dt} + \{\frac{\partial a}{\partial u}(t, \overline{u}) \frac{du}{dt} + \frac{\partial b}{\partial u}(t, \overline{u})\} p \]  \hspace{1cm} (4.2)

and

\[ \psi(p) = \{a(t, \overline{u}+p) - a(t, \overline{u})\} \frac{dp}{dt} + \{a(t, \overline{u}+p) - a(t, \overline{u}) - p \frac{\partial a}{\partial u}(t, \overline{u})\} \frac{du}{dt} \]

\[ + \{b(t, \overline{u}+p) - b(t, \overline{u}) - p \frac{\partial b}{\partial u}(t, \overline{u})\} , \hspace{1cm} (4.3) \]

where \( L \) is the linearization of \( F \) at \( p = 0 \). Note that both operators \( L \) and \( \psi \) are defined for values of \( t \) in the segment \([0, T]\), where \( \overline{u} \) is considered as a known function which belongs to \( C^2[0, T] \). Let us now specify the spaces \( N \) and \( B \):

\[ N = \{p \mid p \in C^2[0, T], \; p(0) = p'(0) = 0\} \]

with

\[ |p| = \max\limits_{[0, T]} |p(t)| + \sqrt{\varepsilon} \max\limits_{[0, T]} |p'(t)| \]  \hspace{1cm} (4.4)

and

\[ B = \{q \mid q \in C^0[0, T]\} \text{ with } \|q\| = \max\limits_{[0, T]} |q(t)| \]  \hspace{1cm} (4.5)

In order to apply the fixed point theorem we have to verify first whether the conditions for \( L \) and \( \psi \) in Section 3 are satisfied. We note already that \( F(0) = 0 \).

From the method of energy integrals we obtain:

\[ |L^{-1}[q]| < C(T) \|q\| , \hspace{1cm} (4.6) \]
for $\varepsilon$ sufficiently small. $C(T)$ is a number depending only on $T$ and not on $\varepsilon$. For a detailed proof we refer the reader to [11], pp. 2-6. In fact the norm (4.4) has been chosen in accordance with this a priori estimate.

A straightforward easy calculation and the assumption that the coefficients $a$ and $b$ belong to $C^3(R_+ \times R)$ yield

$$\|\psi(p_1) - \psi(p_2)\| < C(T) \varepsilon^{-\frac{1}{2}} |p_1 - p_2|$$

(4.7)

for all $p_{1,2}$ with $|p_{1,2}| < \rho$; $C(T)$ denotes a generic constant, dependent of $T$, but independent of $\varepsilon$. It follows that

$m(\rho) = C(T) \varepsilon^{-\frac{1}{2}}$ and hence $m(\rho_0) = C(T) \rho_0 \varepsilon^{-\frac{1}{2}} = \frac{1}{2} \ell = (2C(T))^{-1}$ or

$$\rho_0 = \frac{\varepsilon^{\frac{1}{2}}}{2C(T)^2}$$

(4.8)

The right hand side of equation (2.11), say $f$, has the property

$$\|f\| \leq K \varepsilon \leq \frac{1}{2} \ell \rho_0 = \frac{\varepsilon^{\frac{1}{2}}}{4C(T)^3}$$

(4.9)

for $\varepsilon$ sufficiently small; $K$ denotes again a constant independent of $\varepsilon$. From (4.6)-(4.9) it now follows that all conditions of the fixed point theorem are fulfilled and so we may conclude that there exists a solution $\overline{R}$ of (2.11)-(2.12) with

$$|\overline{R}| \leq 2 \ell^{-1} \|f\| = 2C(T) \|f\| = 0(\varepsilon)$$

and due to (2.10) we have also

$$|\overline{R}| = 0(\varepsilon)$$

(4.10)
Because the solution of the initial value problem (2.11)-(2.12) with specified right hand side is unique in \([0,T]\) the remainder term \(R\) is also uniquely determined.

5. Conclusions

From (2.9) and (4.10) it follows immediately that \(|\tilde{u}-u| = 0(\varepsilon)| or by means of (2.1) and the norm definition (4.4):

\[
u(t) = \omega(t) + \varepsilon \frac{d}{d\tau}(\frac{t}{\varepsilon}) + 0(\varepsilon) = \omega(t) + 0(\varepsilon), \text{ uniformly in } [0,T] \quad (5.1)
\]

and

\[
\frac{d\nu}{dt}(t) = \frac{d\omega}{dt}(t) + \frac{d\nu(t)}{d\tau}(\frac{t}{\varepsilon}) + 0(\sqrt{\varepsilon}), \text{ uniformly in } [0,T] \quad (5.2)
\]

Hence the solution \(\tilde{v}\) of the reduced problem (1.10)-(1.11) is in \([0,T]\) a uniform approximation of the solution \(u\) of the full problem (1.7)-(1.8); the error in \([0,T]\) is of the order \(\varepsilon\).

The derivative of \(u(t)\) is also uniformly approximated by the derivative of \(\omega(t)\) in any segment \([\delta,T]\) with \(\delta>0\) and \(\delta\) independent of \(\varepsilon\); the error is however of the order \(\varepsilon^{\frac{1}{2}}\).

For an approximation of \(\frac{d\nu}{dt}\) in the whole segment \([0,T]\) we need an extra correction \(\frac{d\nu(t)}{d\tau}(\frac{t}{\varepsilon})\), which has only a significant contribution in an \(\varepsilon\)-neighbourhood of \(t = 0\). These results form a satisfactory answer to the question formulated in formula (1.12) of section 1.

Remarks

1. The fixed point theorem of section 3 may also be used for certain singular perturbation problems of elliptic type. We refer the reader to van Harten [7].
2. As already remarked in the Introduction, the theory has been generalized to Cauchy problems for hyperbolic equations, when $L_2$ is a hyperbolic differential operator of the type

$$L_2 = \frac{\partial^2}{\partial t^2} - \sigma^2(x,t) \frac{\partial^2}{\partial x^2},$$

with $\sigma(x,t) \in C^1\{-\infty < x < +\infty, \ 0 \leq t < \infty\}$ and $\sigma(x,t)$ is uniformly bounded in any strip $\{(x,t) \mid -\infty < x < +\infty, \ 0 \leq t \leq s\}$. We refer the reader to lit. [11].

3. An interesting question is to investigate what happens for $t > T$, the region where the solution $w$ of the non linear reduced equation is multivalued. Let us consider as an example:

$$\varepsilon \frac{d^2 u}{dt^2} + \{(u-\sqrt{3})^2 - 1\} \frac{du}{dt} = 1, \ \ 0 < t < \infty \quad (5.3)$$

with $u(0) = u'(0) = 0$. \quad (5.4)

The reduced problem is

$$\{(w-\sqrt{3})^2 - 1\} \frac{dw}{dt} = 1, \ \ 0 < t < \infty \quad (5.5)$$

with $w(0) = 0$. The solution of this initial value problem is given by

$$1/3 \ (w-\sqrt{3})^3 - (w-\sqrt{3}) = t \quad (5.6)$$

The relation between $w$ and $t$ is sketched in the figure.
We expect that the solution $u$ of (5.3)-(5.4) will exhibit a rapid change at $t = T$. For more information about this phenomenon we refer the reader to [14].

REFERENCES


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