ON THE LOCALISATION OF THE ABSOLUTE SUMMABILITY OF THE R-TH DERIVED SERIES OF FOURIER SERIES

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1. Let $S_n$ denote the $n$th partial sum of the series $\sum a_n$ and let $\sigma_n^{(a)}$ and $t_n^{(a)}$ denote the $n$th Cesàro means of order $a$ ($a > -1$) of the sequences $\{S_n\}$ and $\{na_n\}$ respectively. Let $r$ be a positive integer and $T_n^{(r)}$ denote the $n$th Cesàro sum of order $r$ of the sequence $\{na_n\}$, so that

$$t_n^{(r)} = \frac{T_n^{(r)}}{A_n^{(r)}},$$

where

$$A_n^{(r)} = \frac{(n+1)(n+2)\cdots(n+r)}{1\cdot2\cdot3\cdots r}.$$

We write

$$R_n = \frac{1}{\log n} \sum_{v=1}^{n} \frac{S_v}{v}.$$

The series $\sum a_n$, or the sequence $\{S_n\}$, is said to be absolutely summable $(R, \log n, 1)$ or summable $|R, \log n, 1|$, if the sequence $\{R_n\}$ is of bounded variation, that is to say, the infinite series

$$(1.1) \quad \sum |R_n - R_{n+1}|$$

is convergent. When the condition (1.1) is satisfied with \( \{S_n\} \) replaced by \( \{s_n^{(r)}\} \), the series \( \sum a_n \) is said to be summable \(|(R, \log n, 1). (C, r)| \).

We write

\[
\Delta U_n = \Delta U_n = U_n - U_{n+1},
\]

\[
\Delta^n U_n = \Delta \Delta^{n-1} U_n, \quad (r \geq 2).
\]

\( \sum' \) denotes summations over \(-\infty < \nu \leq -1, \ 1 \leq \nu \leq n-1, \) and \( n+1 \leq \nu < \infty \).

The following identities are well-known ([4], [9], [10]):

\[
(1.2) \quad t_n^{(r)} = n(s_n^{(r)} - s_{n-1}^{(r)}),
\]

\[
(1.3) \quad n a_n = \Delta^r T_n^{(r)},
\]

\[
(1.4) \quad \Delta^n U_n = \sum_{\nu=0}^{r} (-1)^{\nu} \binom{r}{\nu} U_{n-\nu}.
\]

Let \( f(t) \) be a periodic function with period \( 2\pi \) and integrable in the Lebesgue sense in \((-\pi, \pi)\). Let the Fourier series associated with the function \( f(t) \) be

\[
\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t).
\]

The allied series of the Fourier series is

\[
\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) \equiv \sum_{n=1}^{\infty} B_n(t).
\]

2.

Regarding the absolute Cesàro summability of the derived series of a Fourier series the following result is known [8].
Theorem A. If
\[ g_1(t) = \frac{1}{t} \int_0^t \frac{f(x+u) - f(x-u)}{u} \, du \]
is of bounded variation in \((0, \pi)\), the derived series of Fourier series of \(f(t)\), at \(t = x\), is summable \(|C, 2 + \delta|\) where \(\delta > 0\).

Since a Lebesgue integral is absolutely continuous, it is plain that \(g_1(t)\) is of bounded variation in any range \((n, \pi)\), \(n > 0\), and therefore it follows that summability \(|C, 2 + \delta|\) of the first derived series of Fourier series depends only upon the behaviour of the generating function in the immediate neighbourhood of the point under consideration. In 1962, the author [11] established that the \(|C, 2|\) summability of the derived series of Fourier series is not necessarily a local property. In this paper we generalise this result by establishing the following.

Theorem 1. The summability \(|(R, \log n, 1)(C, r)|\) of the \(r\)th derived series of the Fourier series of \(f(t)\) is not a local property of the function.

In the following theorem we determine a condition under which the summability \(|(R, \log n, 1)(C, r)|\) of the \(r\)th derived series of Fourier series becomes a local property of the generating function.

Theorem 2. The summability \(|(R, \log n, 1)(C, r)|\) of the \(r\)th derived series of Fourier series depends only on the behaviour of the generating function \(f(t)\) in the immediate neighbourhood of the point \(t = x\), if, when \(r\) is even,
\[
\sum_{n=2}^{\infty} \frac{|A_n(x)|}{n \log n} < \infty,
\]
and when \(r\) is odd
\[
\sum_{n=2}^{\infty} \frac{|B_n(x)|}{n \log n} < \infty.
\]
The following lemmas are pertinent to the proof of our theorems.

Lemma 1. ([12], [15]). If the series \( \sum a_n \) is summable \( |R, \lambda_n, k| \), then \( \sum a_n / \lambda_n^k \) is summable \( |R, e^{\lambda_n}, k|, \ k > 0 \).

Lemma 2. A necessary condition for the series \( \sum a_n \) to be summable \( |(R, \log n, 1). (C, r)| \) is that the series

\[
\sum \frac{|a_n|}{n^{\alpha+1} \log n}
\]

be convergent.

Proof. Since the series \( \sum a_n \) is summable \( |(R, \log n, 1). (C, r)| \) the series \( \sum \{\sigma_n^{(r)} - \sigma_{n-1}^{(r)}\} \) is summable \( |R, \log n, 1| \). Applying Lemma 1 twice it then follows that the series

\[
\sum \frac{\{\sigma_n^{(r)} - \sigma_{n-1}^{(r)}\}}{n \log n}
\]

is summable \( |R, e^n, 1| \), that is, absolutely convergent.* By virtue of the identity (1.2) this is equivalent to

\[
\sum \frac{|t_n^{(r)}|}{n^2 \log n} < \infty.
\]

* It is well-known [7] that the summability \( |R, e^n, k| \) is ineffective in the sense that it sums only convergent series. The analogue of this theorem for absolute Riesz summability in the special case \( k = 1 \) was established independently by Sunouchi [14] and Mohanty [13]. The complete analogue is due to Dikshit [6].
Also, by virtue of the identities (1.3) and (1.4) we have
\[
\sum \frac{|a_n|}{n^{r+1} \log n} = \sum \frac{\Delta^r T_n^{(r)}}{n^{r+2} \log n}
\]
\[
\leq 2^r \sum \frac{|T_n^{(r)}|}{n^{r+2} \log n}
\]
\[
= 0(1) \sum \frac{|t_n^{(r)}|}{n^2 \log n} < \infty.
\]
Hence the lemma.

Lemma 3 [3]. If the series
\[
\sum \frac{|S_n|}{n \log (n+1)}
\]
is convergent, then the sequence \( \{S_n\} \) is summable \(|R, \log n, 1|\).

Lemma 4. If the series
\[
\sum \frac{|o_n^{(r)}|}{n \log n} < \infty
\]
then the series \( \sum a_n \) is summable \(|(R, \log n, 1).(C, r)|\).

The lemma follows clearly by an appeal to Lemma 3.

Lemma 5 [3]. If \( S_n^{(r)} \) denotes the nth Cesàro mean of order \( r \) of the \( r \)th derived series of Fourier series and if \( r \) is even, then
\[
S_n^{(r)}(x) = \frac{2}{\pi} \int_{0}^{\delta} \phi(t) \left\{ \frac{d^r}{dt^r} \left[ \frac{1}{A_n^{(r)}} \sum_{v=0}^{n} \frac{\sin(v+\frac{1}{2})t}{2\sin \left( \frac{t}{2} \right)} \right] \right\} dt
\]
\[
- \frac{4}{\pi} \frac{(n+\frac{1}{2})^r}{A_n^{(r)}} \int_{0}^{\delta} \phi(t) \sin \left( \frac{t}{2} \right) \frac{\sin (n+\frac{1}{2}+\frac{1}{2} r) t}{(2 \sin \left( \frac{t}{2} \right)^{r+2}} dt
\]
\[
+ 0 \left( \sum_{v} \frac{|A_v^{(r)}(x)|}{(n-v)^2} \right) + O\left( |A_n^{(r)}(x)| \right) + O\left( \frac{1}{n} \right).
\]
Lemma 6 [1]. If the series
\[ \sum_{n=1}^{\infty} \frac{|A_n(x)|}{n \log n} \]
is convergent, then the series
\[ \sum_{n=1}^{\infty} \frac{1}{n \log n} \sum_{n=1}^{\infty} \frac{|A_n(x)|}{(n-\nu)^2} \]
is convergent.

Lemma 7 [5]. Suppose \( f_n(x) \) to be measurable in \( (a,b) \), where \( b-a \leq \infty \), for \( n = 1,2,\ldots \), then a necessary and sufficient condition that for every function \( \phi(x) \), integrable over \( (a,b) \), the functions \( f_n(x)\phi(x) \) should be integrable over \( (a,b) \) and
\[
\sum_{n=1}^{\infty} \left| \int_{a}^{b} f_n(x)\phi(x)dx \right| < \infty ,
\]
is that \( \sum_{n=1}^{\infty} |f_n(x)| \) is essentially bounded in \( (a,b) \).

4.

Proof of Theorem 1. Let \( r \) be even. The theorem will be established if we prove that there is a function integrable over \((a,\beta) \subset (0,\pi)\) and zero in the remainder of the interval whose \( r \)th derived series of Fourier series is not summable \(|(R, \log n, 1),(C, r)|\). By virtue of Lemma 2, it is therefore sufficient to show that there exists a function \( x \) integrable over \((a,\beta)\) and such that
\[
\sum_{n=2}^{\infty} \left| \int_{a}^{\beta} x(t) \frac{\cos nt}{n \log n} dt \right| = \infty .
\]
But
\[
\frac{\sum_{n=2}^{\infty} \left| \cos nt \right|}{n \log n} \geq \frac{\sum_{n=2}^{\infty} \cos^2 nt}{n \log n} \geq \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n \log n} - \frac{1}{2} \left| \sum_{n=2}^{\infty} \frac{\cos nt}{n \log n} \right| = \infty ,
\]

53
and therefore the theorem follows by Lemma 7. The proof when \( r \) is odd is similar.

Proof of Theorem 2. Let us suppose that \( r \) is even. By virtue of Lemma 5 we have

\[
S_n^{(r)}(x) = \frac{2}{\pi} \int_0^\delta \phi(t) \left\{ \frac{d^n}{dt^n} \left( \frac{1}{A_n^{(r)}} \sum_{\nu=0}^n A_n^{(r-1)} \frac{\sin (\nu+\frac{1}{2})t}{2\sin \frac{t}{2}} \right) \right\} dt
\]

\[- \frac{4}{\pi} \frac{(n+\frac{1}{2})^r}{A_n^{(r)}} \int_0^\delta \phi(t) \sin \frac{t}{2} \frac{\sin (n+\frac{1}{2}+\frac{1}{2}n)t}{(2\sin \frac{t}{2})^{r+2}} dt
\]

\[+ 0 \left( \sum \frac{|A_{\nu}(x)|}{(n-\nu)^2} \right) + 0(|A_n(x)|) + 0 \left( \frac{1}{n} \right).
\]

\[= \sum_{\nu=1}^5 M_{\nu}, \text{ say.}
\]

We observe that for positive \( \delta \), however small but fixed, the convergence of the series

\[
\sum \frac{|M_{\nu}|}{n \log n} \quad (\nu = 1, 2)
\]

depends only upon the behaviour of the generating function \( f(t) \) in the immediate neighbourhood of the point \( x \). Also, since by the hypothesis of the theorem

\[
\sum \frac{|A_n(x)|}{n \log n} < \infty,
\]

by an appeal to Lemma 4 it follows that for establishing the theorem we need only show that

\[
\sum \frac{1}{n \log n} \sum \frac{|A_{\nu}(x)|}{(n-\nu)^2}
\]
is convergent. But this follows by virtue of Lemma 6 and the hypothesis of the theorem. This completes the proof of the theorem when \( r \) is even. A similar proof holds for the case when \( r \) is odd.

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REFERENCES


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