The triple \((X,T_1,T_2)\) where \(X\) is a set and \(T_1\) and \(T_2\) are topologies on \(X\) is called a bitopological space. Kelly [2] initiated the systematic study of such spaces and several other authors have contributed to the development of the theory. In particular, Pervin [3] introduced connectedness properties for bitopological spaces, and Birsan [1], Reilly [4] and Swart [5] have discussed various aspects of this topic. This note considers quasi-components in bitopological spaces.

Definition 1. (Pervin [3]). A bitopological space \((X,T_1,T_2)\) is pairwise connected if and only if \(X\) cannot be expressed as the union of two non-empty disjoint sets \(A\) and \(B\) such that
\[
(A \cap T_1 \text{cl} A) \cup (T_2 \text{cl} A \cap B) = \emptyset .
\]
(Throughout this paper \(T_1 \text{cl} A\) denotes the \(T_1\) closure of the set \(A\).) If \(X\) can be so expressed we write \(X = A/B\) and this is a separation of \(X\), which is then pairwise disconnected.

In the bitopological space \((X,T_1,T_2)\) we define a relation \(R\) by \((x,y) \in R\) if and only if \(x\) and \(y\) cannot be separated by a separation of \(X\). That is, for each separation \(A/B\) of \(X\) either \(x \in A\) and \(y \in A\) or \(x \in B\) and \(y \in B\). It is easy to see that \(R\) is an equivalence relation on \(X\).

Definition 2. Let \(x\) be a point of \((X,T_1,T_2)\). The equivalence class of \(x\) with respect to \(R\) is called the quasi-component of \(x\), and is denoted by \(Q_x\).

The bitopological quasi-component \(Q_x\) in \((X,T_1,T_2)\) is not related in a simple way to the topological quasi-components of \(x\) in the spaces \((X,T_1)\) and \((X,T_2)\). For example, let \(X = \{a,b\}\).
Let \( T_1 = \emptyset, x, \{ a \} \) and \( T_2 = \emptyset, x, \{ b \} \). Then \( Q_a = \{ a \} \) for \( X = \{ a \}/\{ b \} \) is a separation, but both topological quasi-components of \( a \) are \( X \) since \((X, T_1)\) and \((X, T_2)\) are connected.

The proof of the following proposition is straightforward.

**Proposition.** (a) Every component of \((X, T_1, T_2)\) is contained in a quasi-component.

(b) Every quasi-component of \((X, T_1, T_2)\) is a union of components.

(c) If \( f : (X, T_1, T_2) \to (Y, S_1, S_2) \) is pairwise continuous then the image under \( f \) of each quasi-component of \( X \) lies in some quasi-component of \( Y \).

(d) A quasi-component is a component if and only if it is pairwise connected.

The following concept was introduced by Birsan [1].

**Definition 3.** In the bitopological space \((X, T_1, T_2)\), \( T_1 \) is locally connected with respect to \( T_2 \) if for each point \( x \) in \( X \) and each \( T_1 \) open set \( U \) containing \( X \) there is a pairwise connected \( T_1 \) open set \( G \) such that \( x \in G \subseteq U \).

\((X, T_1, T_2)\) is pairwise locally connected if \( T_1 \) is locally connected with respect to \( T_2 \) and \( T_2 \) is locally connected with respect to \( T_1 \).

Birsan showed that if \( T_1 \) is locally connected with respect to \( T_2 \) then the components of \((X, T_1, T_2)\) are \( T_1 \) open and \( T_1 \) closed.

**Theorem 1.** In a pairwise locally connected bitopological space each quasi-component is a component.

**Proof.** Let \((X, T_1, T_2)\) be pairwise locally connected and \( x \in X \).

Then \( Q_x = \bigcup \{ C_\alpha : \alpha \in A \} \) where the \( C_\alpha \) are distinct components of \((X, T_1, T_2)\). Then \( x \in C_\beta \) for some \( \beta \in A \), and suppose \( y \in Q_x - C_\beta \).

Now since \( C_\beta \) is \( T_1 \) open and \( T_2 \) closed, \( X = C_\beta/X - C_\beta \) is a separation of \( X \) and \( x \in C_\beta \), \( y \in X - C_\beta \). Thus \( y \notin Q_x \) which is a contradiction. Hence \( Q_x - C_\beta = \emptyset \), so that \( Q_x = C_\beta \).
It is an immediate consequence of [4, Definition 4] that the quasi-components of a pairwise totally disconnected space are its points, and hence coincide with the components. Such a space is the set of rational numbers regarded as a subspace of \((\mathbb{R}, L, R)\) where \(R\) is the set of real numbers and \(L\) and \(R\) are the left hand and right hand topologies on \(R\) with bases \(\{(-\infty, x) : x \in \mathbb{R}\}\) and \(\{(x, +\infty) : x \in \mathbb{R}\}\) respectively. This example shows that bitopological quasi-components need not be closed, in contrast to the topological situation. We have, however, the following results.

**THEOREM 2.** If \(x\) is any point of \((X, T_1, T_2)\), \(\{W_\alpha : \alpha \in A\}\) is the family of all \(T_1\) open \(T_2\) closed sets containing \(x\), and \(\{V_\beta : \beta \in B\}\) is the family of all \(T_1\) closed \(T_2\) open sets containing \(x\), then \(Q_x = (\bigcap_{\alpha \in A} W_\alpha) \cap (\bigcap_{\beta \in B} V_\beta)\).

**Proof.** Let \(U = (\bigcap_{\alpha \in A} W_\alpha) \cap (\bigcap_{\beta \in B} V_\beta)\), and suppose \(y \in U\). Then \(y\) belongs to each \(W_\alpha\) and to each \(V_\beta\), and hence \(y\) cannot be separated from \(x\). Thus \(y \in Q_x\) so that \(U \subseteq Q_x\). On the other hand, suppose that \(y \in Q_x\) and \(y \notin U\). Then \(y \notin W_\alpha\) for some \(\alpha \in A\) or \(y \notin V_\beta\) for some \(\beta \in B\). In the first case \(X = W_\alpha / X - W_\alpha\) is a separation separating \(x\) and \(y\), which contradicts \(y \in Q_x\).

Similarly in the second case. Hence \(Q_x \subseteq U\) as desired.

**Corollary.** Any quasi-component \(Q_x\) of a bitopological space \((X, T_1, T_2)\) satisfies the equation \(Q_x = (\bigcap_{\alpha \in A} W_\alpha) \cap (\bigcap_{\beta \in B} V_\beta)\).

**Proof.** With the notation of Theorem 2, \(\bigcap_{\beta \in B} V_\beta\) is a \(T_1\) closed set containing \(Q_x\) so that \(T_1 \text{cl} Q_x \subseteq \bigcap_{\beta \in B} V_\beta\). Similarly \(T_2 \text{cl} Q_x \subseteq \bigcap_{\alpha \in A} W_\alpha\), so that we have \(T_1 \text{cl} Q_x \cap T_2 \text{cl} Q_x \subseteq (\bigcap_{\alpha \in A} W_\alpha) \cap (\bigcap_{\beta \in B} V_\beta) = Q_x\).

Clearly \(Q_x \subseteq (T_1 \text{cl} Q_x) \cap (T_2 \text{cl} Q_x)\) and the equation is satisfied.

**REFERENCES**


117


University of Auckland